Hamilton transversals in tournaments

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§1 Introduction

This is based on a paper by Debsoumya Chakraborti, Jaehoon Kim, Hyunwoo Lee, and Jaehyeon Seo (from 2023). The theorem statement is the following.

Theorem 1.1

Let $T_1, \ldots, T_{n-1}$ be $n-1$ tournaments on a vertex set V of size $|V| = n$, where n is sufficiently large. Then there exists a rainbow Hamilton path — i.e., a Hamilton path such that every edge comes from a distinct T_i.

Definition 1.2. A tournament on V is a directed graph such that there is one directed edge between every pair of vertices (i.e., for all $v \neq v'$, there is an edge between v and v').

This is a natural question to consider, and it's not true if we replace $n-1$ by any smaller number (as a Hamilton path itself has $n-1$ edges, so to get a rainbow Hamilton path, we certainly need at least $n-1$ colors). But in the proofs, it turns out that the fact we have *tournaments* is very strong — every edge is covered in one of two directions by every T_i, and this gives us a lot of choices to choose from. It's unclear what happens if we replace tournaments by more general directed graphs (e.g., what we could replace $n-1$ by), and the authors ask this question in the paper.

§1.1 A version with n colors

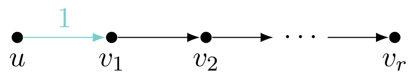
First, Theorem 1.1 is actually easy if we replace $n-1$ by n.

Lemma 1.3

If we have n tournaments $T_1, \ldots, T_n$, then there is a rainbow Hamilton path.

Proof. Let $v_1 \to v_2 \to \cdots \to v_r$ be a rainbow path of maximum length, and assume for contradiction that $r < n$. Then there's at most $n-1$ vertices in the path, so we've used at most 2 colors; and there's n colors in total, so we can suppose we haven't used the colors 1 and 2. Also, we've used at most $n-1$ vertices, so there's at least one untouched vertex; let u be such a vertex.

First, the edge $\overrightarrow{uv_1}$ cannot be in T_1 — if it were in T_1, then we could get a longer path by placing $u \to v_1$ at the beginning (using color 1, which doesn't appear in the rest of the path).



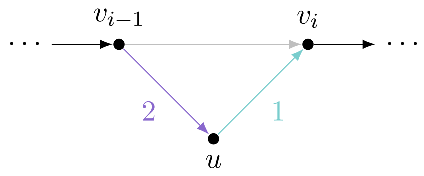
For the same reason, we cannot have $\overrightarrow{uv_1}$ in T_2. This means the edge must be present in the other direction in both T_1 and T_2 — i.e., $\overrightarrow{v_1u}$ is in T_1 and T_2.

Now we'll use induction.

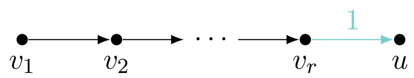
Claim 1.4 — For all $i = 1, \ldots, r$, the edge $\overrightarrow{v_iu}$ is in T_1 and T_2.

Proof. We already know this is true for $i = 1$. Now assume this is not true, and consider the smallest i for which it's not true. This means $\overrightarrow{uv_i}$ is in T_1 or T_2; we can assume it's in T_1.

Meanwhile, because i is the *smallest* such index, we have $\overrightarrow{v_{i-1}u} \in T_2$. This means we can find a longer rainbow path than the one we started with, by taking $v_{i-1} \xrightarrow{2} u \xrightarrow{1} v_i$. □



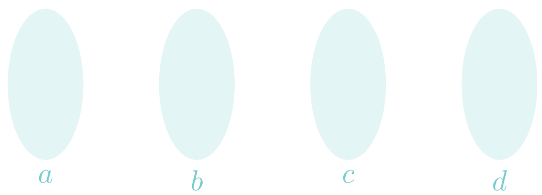
This induction goes all the way up to $i = r$, and that tells us $\overrightarrow{v_ru}$ is in T_1 and T_2. But this means we could insert u at the end of our path (in either color 1 or 2).



So this is a contradiction. □

§1.2 The proof idea

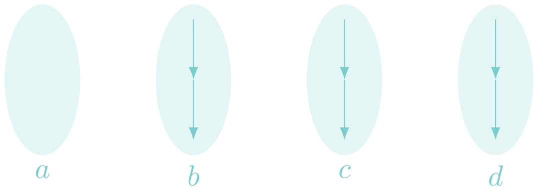
Our goal is to reduce from n colors to $n-1$; the method used for this is absorption. Roughly, what happens is that we first partition the whole vertex set into some smaller parts V_1, V_2, V_3, and V_4, whose sizes we call a, b, c, and d.



Then the rough idea is to find disjoint sets of colors $A, B, C, D \subseteq [n-1]$ such that A is slightly smaller than the corresponding set of vertices, and B, C, and D are slightly larger — specifically,

$$\begin{aligned} |A| &\leq a - \varepsilon n, \\ |B| &\geq b + \varepsilon n, \\ |C| &\geq c + \varepsilon n, \\ |D| &\geq d + \varepsilon n. \end{aligned}$$

Then because $|B| > b$, we can use Lemma 1.3 to find a Hamilton path in V_2 using colors in B, and we can do the same for V_3 (using colors in C) and V_4 (using colors in D).



But since $|B| > b$, we'll have leftover colors; we'll call the sets of these leftover colors B^*, C^*, and D^*. And we know the size of $B^* \cup C^* \cup D^*$, which we call x.

Then using absorption, we'll prove a theorem that looks something like the following: for all $S \subseteq B \cup C \cup D$ with $|S| = x$, there exists a rainbow coloring of all the remaining things using $A \cup S$.

This is kind of the idea of what will happen, though not precisely. But we can see that first proving a statement like Lemma 1.3 is important in this proof — we first assign abstract colors to our substructures using a weaker lemma, and then we have leftovers that we want to absorb into the remaining part.

§2 Proof of Theorem 1.1

Now we'll discuss the details of the proof.

Definition 2.1. For a set of tournaments $\mathbb{T} = \{T_1, \dots, T_m\}$ and some $\alpha \in (0, 1)$, we define

$$\mathbb{T}^\alpha = \{\overrightarrow{uv} \mid \overrightarrow{uv} \text{ lies in at least an } \alpha\text{-fraction of } T_1, \dots, T_m\}.$$

Fact 2.2 — For all $u \neq v$ and all $\alpha < \frac{1}{2}$, either $\overrightarrow{uv}$ or $\overrightarrow{vu}$ lies in $\mathbb{T}^\alpha$.

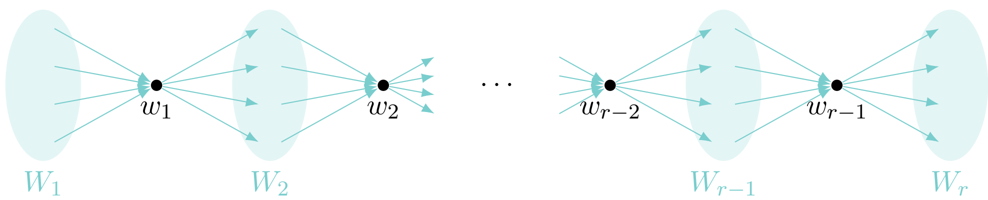
In other words, this means there is a tournament $T \subseteq \mathbb{T}^\alpha$.

§2.1 $\mathbb{H}(\ell, \gamma)$-partitions

In the proof, we'll be partitioning our tournament into smaller pieces; we'll now define the actual form of partitions we'll use.

Definition 2.3. For $\gamma \in [0, 1]$ and $\ell \in \mathbb{N}$, a $\mathbb{H}(\ell, \gamma)$-partition is a tuple $(W_1, \dots, W_r, w_1, \dots, w_{r-1})$ where each W_i is a set of vertices and each w_i an individual vertex, such that:

- All edges from W_i to w_i are present, and all edges from w_i to W_{i+1} are present.
- $V = W_1 \cup \cdots \cup W_r \cup \{w_1\} \cup \cdots \cup \{w_r\}$, and these sets and vertices are disjoint.
- We have $\gamma\ell \leq |W_i| \leq \ell$ for all i.



The first lemma states that we can always find such a partition, for certain parameter settings.

Lemma 2.4

For all $0 \leq \gamma \leq \frac{1}{6}$ and $3 \leq \ell \leq n$, every tournament has a $\mathbb{H}(\ell, \gamma)$-partition.

Proof. The proof is again by induction and contradiction. We induct backwards from $\ell = n$; this case is easy, as we can simply take $W_1 = V$.

Now suppose this statement doesn't hold for ℓ, but it does hold for $\ell+1$; this means any $\mathbb{H}(\ell+1, \gamma)$-partition has a set W_i of size *exactly* $\ell + 1$ (or else it would also be a $\mathbb{H}(\ell, \gamma)$-partition).

Choose a $\mathbb{H}(\ell + 1, \gamma)$-partition that minimizes the number of sets W_i of size $\ell + 1$, and consider one such set W_i. The idea is that we're going to partition W_i into two smaller subparts such that we still have a $\mathbb{H}(\ell + 1, \gamma)$-partition, but with fewer parts of size $\ell + 1$ (violating the minimality of the original partition).

For this, we'll use the following claim.

Claim 2.5 — For every tournament, we have

$$\#\{v \mid \operatorname{indeg}(v) \leq d\} \leq 2d + 1.$$

Proof. Suppose we order the vertices greedily such that at each step, we choose the vertex with maximum in-degree to the remaining vertices. In other words, we pick vertices v_n, v_{n-1}, $\ldots$ such that we choose v_i to maximize the number of edges from $\{v_j \mid j < i\}$ (the set of remaining vertices) to v_i. Then v_i will have in-degree at least $\frac{i-1}{2}$ (since the average in-degree among the i remaining vertices at this step is $\frac{i-1}{2}$). So all vertices v_n, $\ldots$, v_{2d+2} will have in-degree greater than d, which means we're left with at most $2d+1$ vertices whose in-degree is at most d. $\square$

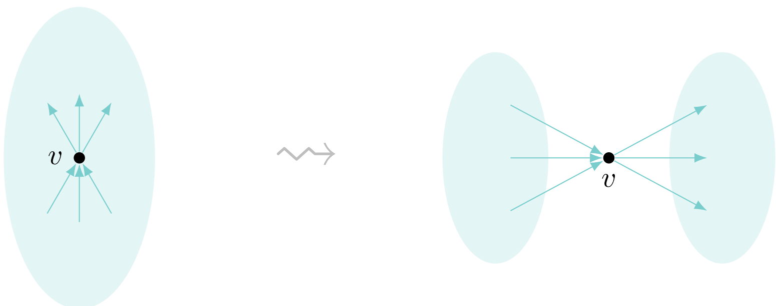
We can do the same for out-degree — i.e., we have

$$\#\{v \mid \operatorname{outdeg}(v) \leq d\} \leq 2d + 1.$$

Now taking $d = \frac{\ell}{6}$, we get that there must exist some $v \in W_i$ that has both big in-degree and big out-degree inside W_i, specifically

$$\operatorname{indeg}_{W_i}(v), \operatorname{outdeg}_{W_i}(v) \geq \frac{\ell}{6} + 1.$$

Then we can partition W_i into two smaller parts — the vertices with edges into v, and the vertices with edges out from v.



So we get a new partition where we keep all sets other than W_i the same, but split W_i into these two sets (with v as the vertex between them). Both sets have size at least $\frac{\ell}{6} + 1$ and strictly less than $\ell + 1$, contradicting the minimality of our partition (i.e., that it had the smallest number of sets of size $\ell + 1$). $\square$

§2.2 The absorption lemma

Now we'll state the actual absorption lemma that we'll use. We'll first state a lemma that applies to a more general setting with bipartite graphs, which we'll deduce our lemma from.

Lemma 2.6 (General absorption)

Let $\alpha \in (0, 1)$, and let H be a bipartite graph on vertex set $A \cup B$ such that:

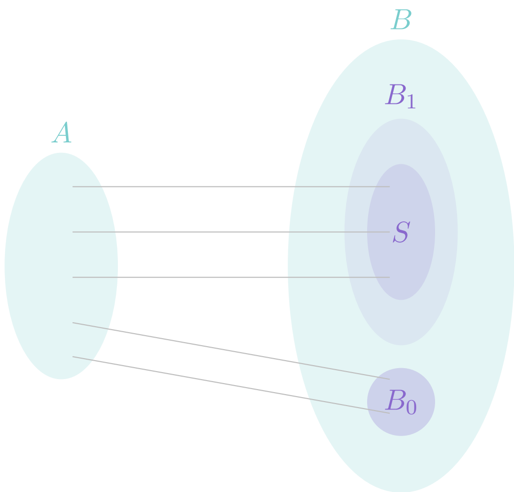
(1) We have $\alpha^2 |B| \geq 8 |A|$.

(2) For all $a \in A$, we have $\deg(a) \geq \alpha |B|$.

Also let $\ell \leq 10^{-5} \cdot \alpha^7 |A|$. Then we can find two disjoint sets $B_0, B_1 \subseteq B$ with

$$|B_0| = |A| - \ell \quad \text{and} \quad |B_1| \geq 10^{-5}\alpha^7 |B|$$

such that for every $S \subseteq B_1$ with $|S| = \ell$, there is a perfect matching between A and $B_0 \cup S$.



This is a general absorption statement, and was actually proved in an earlier paper about colorful spanning trees ('Transversal factors and spanning trees' by Montgomery–Müyesser–Pehova). We'll prove this at the end if we have time, but for now we'll talk about how to use it. If we apply this to our setting, it immediately translates to the following lemma (there are a bunch of parameters that will make sense later on).

Lemma 2.7

Let $0 < \frac{1}{n} \ll \gamma \ll \beta \ll \alpha \leq \frac{1}{2}$, and let $\mathbb{T}$ be a collection of tournaments with $|\mathbb{T}| = m \geq \alpha n$ on a vertex set V of size $|V| = n$. Let $H \subseteq \mathbb{T}^\alpha$ be such that

$$\beta n \leq e(H) \leq \left(\beta + \frac{\gamma}{2}\right) n.$$

Then there exist two disjoint sets of colors $A, C \subseteq [m]$ such that $|A| = e(H) - \gamma m$ and $|C| \geq 10\beta m$, such that for all $C' \subseteq C$ with $|C'| = \gamma m$, there is a rainbow coloring of H using colors in $A \cup C'$.

(The notation $\gamma \ll \beta$ here means that γ is sufficiently small with respect to β.)

Proof. We create a bipartite graph J with vertex sets $E(H)$ (the set of edges in H) on the left, and the set of colors $[m]$ on the right; in particular, these two sets have sizes $e(H)$ and m, respectively. We draw an edge between an edge $\overrightarrow{uv}$ on the left and a color i on the right if $\overrightarrow{uv} \in T_i$. Then we can verify the conditions of Lemma 2.6:

(1) Every edge $\overrightarrow{uv} \in E(H)$ is incident to at least an α-fraction of our colors by the fact that $H \subseteq \mathbb{T}^\alpha$, so $\deg_J(\overrightarrow{uv}) \geq \alpha m$ for all $\overrightarrow{uv}$.

(2) The fact that $\alpha^2 m \geq 8e(H)$ just follows from the way we set the parameters — we prescribed that β is sufficiently small compared to α, and m and n differ by a factor of at most α.

(3) Finally, we have $\ell = \gamma m$, so we need to check that this is at most $10^{-5}\alpha^7 e(H)$. This is again true because γ is sufficiently small compared to α and β.

So we can directly apply Lemma 2.6, which gives this result. □

§2.3 Hamilton paths with prescribed colors

The last lemma we'll need is the following, which states that if we have enough tournaments, then we can find a rainbow Hamilton path that uses a prescribed color.

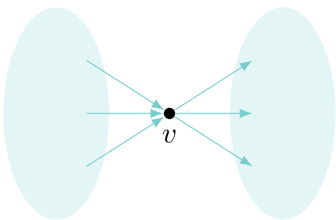
Lemma 2.8

Let $\mathbb{T}$ be a collection of tournaments on $n \geq 4$ vertices, with $|\mathbb{T}| \geq 2n$. Then for every $T_i \in \mathbb{T}$, there exists a rainbow Hamilton path that uses an edge from T_i.

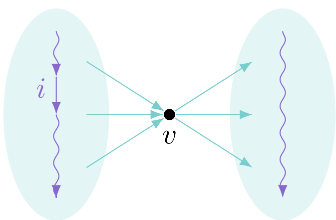
Proof. The proof is again some sort of induction and contradiction. First, we can assume without loss of generality that $|\mathbb{T}| = 2n$. Imagine that we check the small cases $n = 4, 5, 6, 7$ by hand. (The paper doesn't actually do this — they start the induction at $n = 3$ and have an exceptional case where $\mathbb{T}$ is a collection of 'generalized triangles,' but this is somewhat more annoying to state.)

Consider a counterexample $(\mathbb{T}, i)$ with the smallest value of n, and fix some subtournament $T \subseteq \mathbb{T}^{1/2}$.

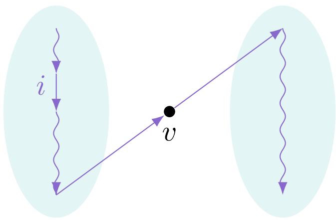
We first pick some v with $\text{outdeg}(v) \geq 4$ (where $\text{outdeg}(v)$ refers to its degree in T), and consider the sets of vertices with edges into v and the set of vertices with edges out from v.



Then for the vertices on the left, we can find a Hamilton path using color i by the induction hypothesis. Meanwhile, for the vertices on the right, we can use Lemma 1.3 to find a rainbow Hamilton path (not using any of the colors that we used on the left).



Then we've only used $n - 2$ colors so far. But every edge in T appears in at least n colors (since $T \subseteq \mathbb{T}^{1/2}$), so we can find some unused colors to link the path on the left to v and v to the path on the right.

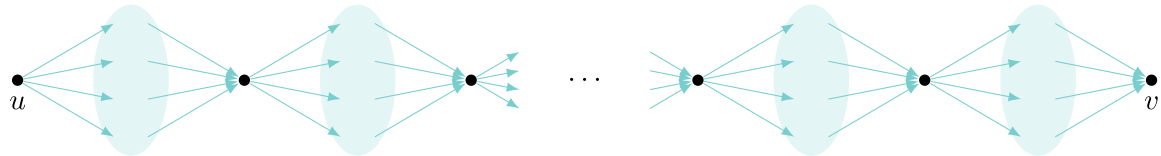


This gives a Hamilton path using color i, as desired. $\square$

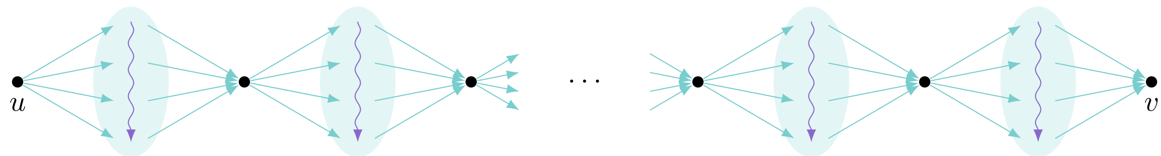
Corollary 2.9

Let $\mathbb{T}$ be a collection of tournaments of size $|\mathbb{T}| = m \geq 4n$ on a vertex set V of size n. Let $B \subseteq [m]$ be a subset of colors of size $|B| \leq \frac{n}{25}$. Let u and v be two vertices such that for all $w \neq u, v$, both $\overrightarrow{uw}$ and $\overrightarrow{wv}$ are in the tournaments corresponding to at least two colors in B. Then there is a rainbow Hamilton path from u to v that uses all colors in B.

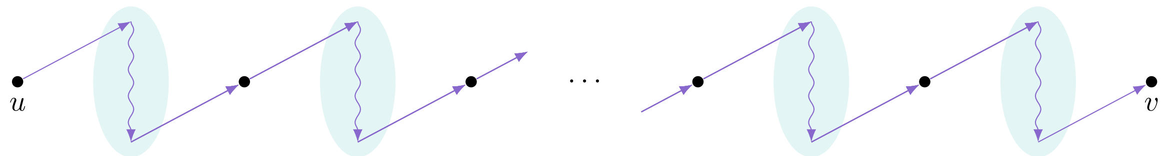
Proof. Let $T \subseteq \mathbb{T}^{1/2}$ be a tournament (so every edge in T is still incident with $2n$ colors), and let $V' = V \setminus \{u, v\}$. Then we can restrict our tournaments to V' and construct a partition as in Lemma 2.4, with u and v on the two ends.



More precisely, we take a $\mathbb{H}(24, \frac{1}{6})$-partition of T' (restricted to V'), which we know exists by Lemma 2.4. Then each set has constant size, so there's linearly many sets. This means we can assign one color in B to each of these sets — if $B = \{1, \ldots, r\}$, then we use Lemma 2.8 to find a rainbow Hamilton path in W_1 using color 1, one in W_2 using color 2, and so on.



And we started with $4n$ colors, so after finding these paths in W_1, …, W_r and using up B, we still have $3n$ colors left. And every edge in T is incident to at least $2n$ colors, so we can greedily pick colors for the remaining edges needed to link up these paths.



(Where did we use the condition about $\overrightarrow{uw}$ and $\overrightarrow{wv}$ being in at least two colors in B? This might have been used for being able to put u and v at the ends of the partition.) $\square$

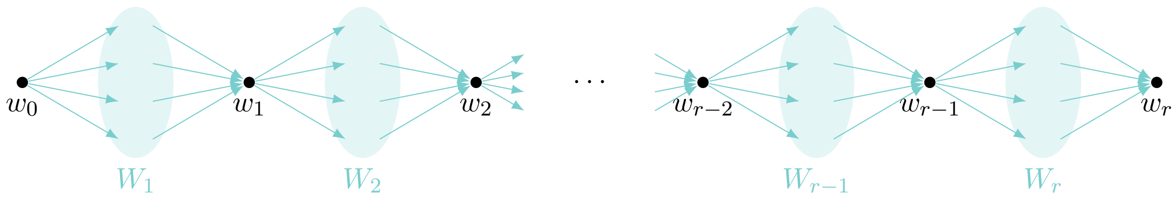
§2.4 Proof outline of Theorem 1.1

Finally, we'll sketch the proof of Theorem 1.1. We'll actually prove the following lemma.

Lemma 2.10

Let $0 < \frac{1}{n} \ll \mu \ll \gamma \ll \alpha \le 1$. Let $\mathbb{T}$ be a collection of tournaments with $|\mathbb{T}| = n-1$, and let $T \subseteq \mathbb{T}^\alpha$ be a subtournament. Let w_0 and w_r be two vertices, and suppose that $(W_1, \ldots, W_r, w_1, \ldots, w_{r-1})$ is a $\mathbb{H}(\mu n, \gamma)$-partition of T restricted to the remaining vertices, such that $\overrightarrow{w_0 u} \in T$ for all $u \in W_1$ and $\overrightarrow{v w_r} \in T$ for all $v \in W_r$. Then there exists a rainbow Hamilton path from w_0 to w_r.

So we're again imagining a setup where we have a partition and two vertices w_0 and w_r at its ends (here the partition is given to us), and we find a Hamilton path in this setup.



This almost implies Theorem 1.1 — to prove Theorem 1.1, we take a slightly larger partition (without vertices on the two ends; we know such a partition exists by Lemma 2.4), then take Hamilton paths in the first and last components (greedily) and apply this lemma on the remainder.

Proof. We'll eventually partition our graph into four sets of edges. For some $1 \le t < \tau \le r$, we'll define:

- E_A as the set of edges in $W_1, \ldots, W_t$.
- E_B as the set of edges in $W_{t+1}, \ldots, W_{\tau-1}, W_{\tau+1}, \ldots, W_r$.
- E_C as the set of edges in W_τ.
- E_D as the set of edges between w_{i-1} and W_i, and W_i and w_{i+1}.

We'll also partition our colors into sets A, B, C, and D that roughly correspond to these edge sets. First, for D, note that since ℓ is linear in n, the number r of components is *constant*. And D is the colors we'll be using for edges between adjacent components, so we can choose D such that:

- D is very small, specifically $|D| \le \mu^2 n$.
- Every edge in E_D is incident to at least $4r$ colors in D.

(This is possible because r is a constant, so we can do this by randomly choosing D or something similar.)

We want some absorption to happen, and this will happen between A and C. So we'll use Lemma 2.7 to take A and C such that

$$|A| = |E_A| - \gamma n \quad \text{and} \quad |C| \ge 10\beta n,$$

and such that for all $C' \subseteq C$ with $|C'| = \gamma n$, there is a rainbow coloring of E_A using $A \cup C'$.

Our goal is to color everything so that in the end, we're left with some $C' \subseteq C$ that we can absorb into A. But we also have to color the edges in B. We'll eventually do this using Lemma 1.3, but that means B can't have the exact same size as E_B — we'll have to have a few more colors (specifically, an extra color for each component) so that we have one extra color and can use Lemma 1.3. But then the problem is that we'll also be left with some colors in B, and B does *not* have the nice property of being absorbed by A. So to deal with this, we'll need C to absorb B first, and then we'll have some leftovers in C that get absorbed by A.

How do we do this? This is what will happen in the area around our special set W_τ. We'll have some colors B^* and D^* left over from B and D, and then we'll use Lemma 2.8 to rainbow-color W_τ (and maybe its boundary edges) with colors in $B^* \cup C \cup D^*$, such that all the colors in $B^* \cup D^*$ get used.

And after this step, we've used all the colors in B and D, so we're left with some $C' \subseteq C$. And then we can use the outcome of Lemma 2.7 to color E_A with $A \cup C'$. $\square$