Flat Littlewood Polynomials Exist

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§1 Introduction

(This talk is based on a paper by Ballister, Bollobás, Morris, Sahasrabudhe, Tiba from 2020.)

Definition 1.1. A polynomial is *Littlewood* if it is of the form $\sum_{k=0}^{n} \varepsilon_k z^k$ for $\varepsilon_k \in \{-1, 1\}$ — i.e., if all its coefficients are ± 1.

Note that for a Littlewood polynomial P, we have

$$\frac{1}{2\pi} \int_0^{2\pi} \left| P(e^{i\theta}) \right|^2 \, d\theta = \sum_{k=0}^{n} |\varepsilon_k|^2 = n + 1.$$

This means (by an averaging argument) that for every Littlewood polynomial P, there exists a point z_1 on the unit circle S^1 with $|P(z_1)| \leq \sqrt{n+1}$, as well as a point $z_2 \in S^1$ with $|P(z_2)| \geq \sqrt{n+1}$.

Question 1.2. For a Littlewood polynomial P, how does the value of $|P(z)|$ vary on the unit circle?

Theorem 1.3

There exist universal constants $\delta, \Delta > 0$ such that for every $n \geq 2$, there exists a Littlewood polynomial P of degree n such that for all $z \in S^1$ we have

$$\delta\sqrt{n} \leq |P(z)| \leq \Delta\sqrt{n}.$$

Polynomials satisfying the above property (i.e., that $\delta\sqrt{n} \leq |P(z)| \leq \Delta\sqrt{n}$ on the unit circle) are called *flat*.

Remark 1.4. Erdős asked whether such polynomials exist, and Littlewood conjectured that they do; this theorem answers both questions.

Remark 1.5. The difficult part of the theorem is the lower bound — polynomials satisfying the upper bound $|P(z)| \leq \Delta\sqrt{n}$ have been known for a while, while polynomials satisfying the lower bound were not previously known (even if we do not require them to also satisfy the upper bound).

Very roughly, the structure of the proof is as follows — we'll let $z = e^{i\theta}$ for $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. We'll decompose P into its real and imaginary part, and work with the two parts separately. To obtain the lower bound on $|P(z)|$, we'll show that the real part of P is large (in magnitude) everywhere except for when θ is in a collection of 'well-behaved' exceptional intervals in $[0, 2\pi)$, and the imaginary part of P is large on these well-behaved intervals. Finally, to obtain the upper bound, we will show that both the real and imaginary parts of P are always $O(\sqrt{n})$.

Now we'll see an outline of the proof with more details.

§1.1 Proof Outline

First, without loss of generality we may assume that n is sufficiently large — then we can adjust δ and Δ to make the statement true for the finitely many remaining n. (For this to work, we just need to be able to find a polynomial with no zeros on the unit circle; the polynomial $1 - z - z^2 - \cdots - z^n$ is such a polynomial for all $n \geq 2$.)

Instead of considering polynomials $P(z)$ of the form $\sum_{k=0}^{n} \varepsilon_k z^n$, we'll find $P(z)$ of the form $\sum_{k=-2n}^{2n} \varepsilon_k z^k$. Then multiplying by z^{2n} (which doesn't affect the magnitudes on the unit circle) gives a polynomial of the original form. (This construction only gives Littlewood polynomials whose degree is a multiple of 4, but we can add on a few extra terms to get any degree.)

We'll use C to denote the subset of $[2n]$ such that for all $k \in C$ we have $\varepsilon_k = \varepsilon_{-k}$, and for all $k \in [2n] \setminus C$ we have $\varepsilon_k = -\varepsilon_{-k}$. Then we can rewrite $P(e^{i\theta})$ as

$$P(e^{i\theta}) = \varepsilon_0 + 2 \sum_{k \in C} \varepsilon_k \cos(k\theta) + 2i \sum_{k \notin C} \varepsilon_k \sin(\theta).$$

We'll define $c(\theta) := \sum_{k \in C} \varepsilon_k \cos(k\theta)$ and $s(\theta) := \sum_{k \notin C} \varepsilon_k \sin(k\theta)$ as the first and second sums.

Our polynomial will have the property that $C \subseteq 2[n]$ (i.e., all elements of C are even), so that $[2n] \setminus C$ contains all the odd numbers, and we can write $[2n] \setminus C = \{1, 3, \ldots, 2n-1\} \cup (2[n] \setminus C)$. We'll decompose s into its parts coming from odd and even k — define $s_o(\theta) = \sum_{k \text{ odd}} \varepsilon_k \sin(k\theta)$ and $s_e(\theta) = \sum_{k \in 2[n] \setminus C} \varepsilon_k \sin(k\theta)$. Then we have decomposed P as

$$P(e^{i\theta}) = \varepsilon_0 + 2c(\theta) + 2is_O(\theta) + 2is_e(\theta).$$

Note that these three parts are unrelated — once we have chosen the set C, each of $c(\theta)$, $s_o(\theta)$, and $s_e(\theta)$ is determined by a disjoint set of coefficients. So we can choose these three parts separately. We'll choose them so that the following properties hold:

(0) Each of $|c(\theta)|$, $|s_o(\theta)|$, and $|s_e(\theta)|$ is $O(\sqrt{n})$ for all θ (and therefore the upper bound holds).

(1) We have $|c(\theta)| \geq \delta\sqrt{n}$ for all $\theta \in [0, 2\pi)$ except for a 'nice' collection of intervals.

(2) On the nice collection of intervals we have $|s_o(\theta)| \geq \delta_{\text{big}}\sqrt{n}$ (where δ_{big} is a reasonably large constant).

(3) We have that $|s_e(\theta)|$ is small for all θ (where 'small' means on the order of $\sqrt{n}$, with a constant less than δ_{big}).

The two most difficult parts of the proof are (1) and (2). We'll do (1) by constructing $c(\theta)$ explicitly, and (2) by using *discrepancy* to prove the existence of such a $s_o(\theta)$.

§1.2 Nice Intervals

Steps (1) and (2) both involve a 'nice' collection of intervals; we'll now make explicit what this means.

Definition 1.6. We say that a collection of intervals $\mathcal{I} = \{I_1, \ldots, I_r\}$ (with $I_i \subseteq \mathbb{R}/2\pi\mathbb{Z}$ for all i) is *nice* if it satisfies the following conditions:

(a) The endpoints of each interval are integer multiples of π/n (i.e., the intervals are 'rational').

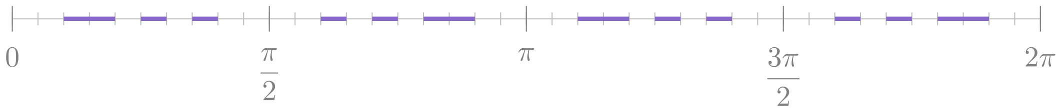
(b) For all $i \neq j$, we have $d(I_i, I_j) \geq \pi/n$ (i.e., the intervals are 'separated').

(c) For all i, we have $|I_i| \leq 6\pi/n$ (i.e., the intervals are 'small').

(d) We have $\mathcal{I} = \pi - \mathcal{I} = \pi + \mathcal{I}$ (i.e., the intervals are 'symmetric').

(e) All our intervals are far away from integer multiples of $\pi/2$ — for all i and all $m \in \mathbb{Z}$ we have

$$I_i \cap \left[\frac{\pi m}{2} - \frac{100\pi}{n}, \frac{\pi m}{2} + \frac{100\pi}{n}\right] = \emptyset.$$



We'll seek a nice collection of intervals such that outside these intervals $c(\theta)$ is large, and inside these intervals $s_o(\theta)$ is large.

§2 Step (1) — Constructing $c(\theta)$

We'll first do Step (1). In this part, we'll also see why the upper bound has been known for a long time (i.e., that there exist Littlewood polynomials with $|P(z)| \le \Delta\sqrt{n}$ on the unit circle) — we'll see an explicit family of polynomials which satisfy the upper bound.

Definition 2.1. The *Rudin–Shapiro polynomials* are defined inductively in the following way: define $P_0 = Q_0 = 1$, and inductively define

$$P_{t+1}(z) = P_t(z) + z^{2^t} Q_t(z)$$
$$Q_{t+1}(z) = P_t(z) - z^{2^t} Q_t(z).$$

One can check (using induction) that P_t and Q_t are both Littlewood polynomials of degree $2^t - 1$, and

$$P_t(z)P_t(1/z) + Q_t(z)Q_t(1/z) = 2^{t+1}.$$

This latter property is useful because it means that for $z \in S^1$, we have

$$|P_t(z)|^2 + |Q_t(z)|^2 = 2^{t+1}.$$

This in particular means that $|P_t(z)|, |Q_t(z)| \le \sqrt{2^{t+1}}$ for all $z \in S^1$, and therefore the polynomials P_t and Q_t themselves satisfy the upper bound in the theorem (i.e., their magnitude is bounded by roughly the square root of their degree). (These polynomials, and the fact that they satisfy the upper bound, have been known before this work.)

We'll now use the polynomials P_t and Q_t to define $c(\theta)$ (in fact, later we'll use them to define $s_e(\theta)$ as well).

Definition 2.2. Let $z = e^{2i\theta}$ and $T = 2^{t+10}$. We then define

$$c(\theta) = \mathrm{Re}(z^T P_t(z) + z^{2T} Q_t(z)).$$

The multiplication by z^T and z^{2T} shifts the exponents we obtain from P_t and Q_t so that they are disjoint (this is why we want T to be around 2^t, and the extra 10 will be useful when we use discrepancy), and therefore this expression will be of the form we want (i.e., a sum of terms of the form $\pm\cos(2k\theta)$, where k ranges over the exponents of z appearing in $z^T P_t(z)$ and $z^{2T} Q_t(z)$) — the set C (of indices where $c(\theta)$ has nonzero coefficient) will be $2C'$ where $C' = \{T, \dots, T + 2^t - 1\} \cup \{2T, \dots, 2T + 2^t - 1\}$. (We will eventually take n to be approximately 2^t, with some adjustments..)

We want to show that we have $|c(\theta)| \ge \delta\sqrt{n}$ everywhere (for some δ) except for a nice collection of intervals. In other words, we want to prove the following statement:

Lemma 2.3

Partition $[0, 2\pi) = I_1 \cup \dots \cup I_{2n}$ into $2n$ equal intervals (i.e., $I_i = [2\pi(i-1)/n, 2\pi i/n)$ for each i). Then

$$\mathcal{I} = \{I_i \mid \text{there exists } \theta \in I_i \text{ with } |c(\theta)| < \delta\sqrt{n}\}$$

is a nice collection of intervals.

Some of the conditions we need to check are clear from the definition — (a) is clearly true; (b) will be satisfied if we combine adjacent unions; (d) is true because $c(\theta)$ is a function of $\cos(2\theta)$, which has the symmetry properties described in (d); and (e) can be verified by direct calculation (since when θ is close to an integer multiple of $\pi/2$, we know z is close to ± 1 or $\pm i$, and in these cases we can compute the relevant quantities and check that the statement holds).

So we only need to verify (c). In order to do this, we will show that $|c(\theta)|$ is frequently large on an interval of reasonable length. To do so, we will rewrite c as

$$c(\theta) = 2^{(t+1)/2}\,\mathrm{Re}(H(2T\theta))$$

for a certain function H (depending on P_t and Q_t) — explicitly, we define

$$H(x) = e^{ix}\alpha(x) + e^{2ix}\beta(x),$$

where $\alpha(x) = 2^{-(t+1)/2}P_t(e^{ix/T})$ and $\beta(x) = 2^{-(t+1)/2}Q_t(e^{ix/T})$. The lemma we'll use to obtain (c) is the following.

Lemma 2.4

Let $0 < \eta < 2^{-11}$. Then for every interval I of length 7η, there exists a subinterval $J \subseteq I$ of length η such that $|\mathrm{Re}(H(x))| \geq \eta^3/2^7$ for all $x \in J$.

Once we know this lemma, we can show that we must have intervals I_i where $|c(\theta)|$ isn't too small once in a while, which will give (c).

Proof Sketch. One can first show that for every x, at least one of $\left|\mathrm{Re}(H^{(k)}(x))\right|$ is large for $k = 0, 1, 2, 3$ (i.e., we consider the function H itself and its first three derivatives, and at every point at least one of these four terms must be large). This will in particular imply that H cannot be small and remain small for a very long time.

Then we split I evenly into seven pieces $I = I_0 \cup \cdots \cup I_6$ and assume for contradiction that there exist $x_0 \in I_0, \ldots, x_6 \in I_6$ such that $|\mathrm{Re}(H(x_i))|$ is small for each i. This will eventually contradict Bernstein's inequality. □

Assuming the lemma (which we won't go into more detail over for the sake of time), we're done with (1) — we've successfully constructed $c(\theta)$ and shown that it's large everywhere except a nice collection of intervals.

§3 Step (2) — Constructing $s_0(\theta)$

Now fix $\mathcal{I}$ to be the collection of nice intervals obtained from (1). We want to show that there exists s_o such that $|s_o(\theta)| \geq \delta_{\mathrm{big}}\sqrt{n}$ on all of $\mathcal{I}$.

We'll do this by choosing s_o such that $|s_o(\theta)|$ is close to the indicator function $\sum_{I\in\mathcal{I}} \mathbf{1}_I \cdot \sqrt{n}$, or possibly a scaled multiple of it (where $\mathbf{1}_I$ is the indicator function of the interval I). Note that this condition still leaves us free to choose the signs of the values of s_θ. So we'll get to choose two things:

- First we'll try to find an assignment of signs to each interval — which we'll view as a map $\alpha\colon \mathcal{I} \to \{\pm 1\}$. We'll choose α to be *symmetric* in the sense that $\alpha(I) = \alpha(\pi - I) = -\alpha(\pi + I)$.
- Given an assignment α, we'll try to find an assignment of coefficients $\varepsilon_1, \varepsilon_3, \ldots, \varepsilon_{2n-1} \in \{-1, 1\}$ such that the associated polynomial $s_o(\theta)$ corresponding to these coefficients is close to the function $\sum_{I\in\mathcal{I}} \alpha(I)\mathbf{1}_I\sqrt{n}$ (or a scaled multiple of it).

We will do this using *discrepancy* — we'll use discrepancy twice, first to select α and then to select the ε_i. In order to make this argument work, we'll first define a few objects that are sort of 'approximate' versions of the ones we'll eventually construct.

Definition 3.1. For every symmetric $\alpha\colon \mathcal{I} \to \{\pm 1\}$, we define *approximate coefficients* for all odd k as

$$\hat{\varepsilon}_k = 2^7\sqrt{n} \cdot \int_{-\pi}^{\pi} \sum_{I \in \mathcal{I}} (\alpha(I)\mathbf{1}_I(\theta)) \sin(k\theta)\, d\theta.$$

(We'll eventually want these to be close to our actual coefficients $\hat{\varepsilon}_k$.)

Definition 3.2. For every symmetric α, we define $\hat{s}_\alpha(\theta) = \sum_{k \text{ odd}} \hat{\varepsilon}_k \sin(k\theta)$.

Roughly, the idea is that we expect $\hat{s}_\alpha(\theta)$ to be close to $\pi \cdot 2^7\sqrt{n}\sum_{I\in\mathcal{I}} \alpha(I)\mathbf{1}_I(\theta)$ (this has to do with the fact that the terms $\hat{\varepsilon}_k$ are sort of defined as the Fourier coefficients of this function), and we'll try to choose the ε_k to be close to the $\hat{\varepsilon}_k$, so that then $s_o(\theta)$ is close to this function as well.

Let $g_\alpha(\theta) = \sum_{I\in\mathcal{I}} \alpha(I)\mathbf{1}_I(\theta)$ for each α. The argument consists of three steps:

(1) First, we will show that there exists some α such that all the associated $\hat{\varepsilon}_k$ are in $[-1, 1]$.

(2) Then we will show that we can find actual coefficients $\varepsilon_k \in \{\pm 1\}$ such that $\hat{s}_\alpha$ and s_o are close everywhere — i.e., such that $|\hat{s}_\alpha(\theta) - s_o(\theta)|$ is small everywhere.

(3) Finally, we will show that for *any* α we have $|\hat{s}_\alpha(\theta)| \geq \delta_{\text{big}}\sqrt{n}$ on all of $\mathcal{I}$.

Both (1) and (2) are applications of discrepancy. (We won't see the details for the sake of time.)

Remark 3.3. It might be surprising at first that we construct s_o using discrepancy, since discrepancy is generally about making things small — e.g., the original problem involved starting with a collection of $\{0,1\}$-vectors and finding a $\{\pm 1\}$-vector with small inner product to all of them — and here we're trying to make s_o large, not small. But the above outline shows how we've reduced the problem to making things small — we want the $\hat{\varepsilon}_k$ to be small and s_o to be close to $\hat{s}_\alpha$ — so it now sounds a lot more like something that can be done using discrepancy.

Meanwhile, (3) is where we use the niceness of $\mathcal{I}$. (It doesn't depend on the specific α we chose, and should hold for all α.) The point is roughly that in our definition of $\hat{\varepsilon}_k$ we have a $\sin(k\theta)$ term, and then in the definition of $\hat{s}_\alpha(\theta)$ we're multiplying by another $\sin(k\theta)$; so we sort of have a $\sin^2$ term, which we'd expect to have a reasonably large contribution to $\hat{s}_\alpha(\theta)$. (The argument involves using a trigonometric identity to estimate $\int_\theta \sum_j \sin((2j+1)\theta_0)\sin((2j+1)\theta)$ or a similar expression.)

§4 Step (3) — Constructing $s_e(\theta)$

Finally, for part (3), we'll define s_e explicitly in the following way — let $z = e^{2i\theta}$, and define

$$s_e(\theta) = \operatorname{Im}(P_{<n+1}(z) - z^T P_t(z) - z^{2T} P_t(z)),$$

where $P_{<n+1}$ denotes the degree n polynomial that agrees with P_{n+1} on the first $n+1$ terms (i.e., the polynomial obtained from P_{n+1} by only considering the terms of degree at most n; note that the polynomials P_t all share the same initial segments). The subtraction of $z^T P_t$ and $z^{2T} P_t$ is to ensure that the terms which appear in $s_e(\theta)$ are precisely those which do not appear in $c(\theta)$.

We define n by taking $2^{-43} < \gamma < 2^{-40}$ and letting n be such that $\gamma n = 2^{t+11} + 2^t - 1$. This means n is quite a bit bigger than T — it's around $2^{t+11}/\gamma \approx 2^{t+50}$.

We want to show that $|s_e(\theta)|$ is always small — we just need it to be small compared to our bound on $|s_o(\theta)|$ on $\mathcal{I}$, so that it doesn't cancel out the contribution from s_o. We can just do this naively (using the $\sqrt{n}$ bound on P_t we saw when constructing $c(\theta)$ earlier) — this gives us a $\sqrt{n}$ upper bound on $|s_e(\theta)|$ where the constant is on the order of 1, while our construction of s_o gave a $\sqrt{n}$ lower bound on $|s_o(\theta)|$ with a much bigger constant (on the order of 2^7).