Asymptotic Properties of Maximal p-Core p'-Partitions

Sanjana Das

July 12, 2022

Partitions

Definition

A *partition* λ of n is a way of writing

$$n = \lambda_1 + \lambda_2 + \cdots + \lambda_k,$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ are positive integers.

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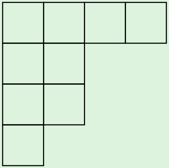
Example

The partitions of 4 are (4), $(3, 1)$, $(2, 2)$, $(2, 1, 1)$, and $(1, 1, 1, 1)$.

Young Diagrams

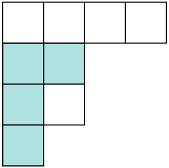
Example

The partition $(4, 2, 2, 1)$ of 9 corresponds to the following Young diagram:



Hook Lengths

Given a box in a Young diagram, its *hook* is the set of boxes below it and to its right (including the square itself):



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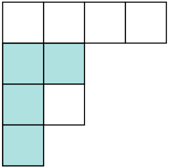
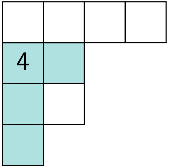


The *hook length* of a box is the number of boxes in its hook:



Representation Theory of S_n

Fact

There is a natural way to index irreducible representations of S_n over $\mathbb{C}$ by partitions of n.

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Character Tables

The **character table** takes each irreducible representation ρ and each conjugacy class, and records their *traces*.

	(1)	(12)	(123)
χ_0	1	1	1
χ_1	1	-1	1
χ_2	2	0	-1

			
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S_3 case

Zeros in the Character Table of S_n

Open Question

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Theorem (McSpirit–Ono)

For each $d > 0$ we have

$$\lim_{n \to \infty} \frac{Z(n)}{p(n)n^d} = +\infty.$$

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Remark

They used p-core p'-partitions to obtain this result.

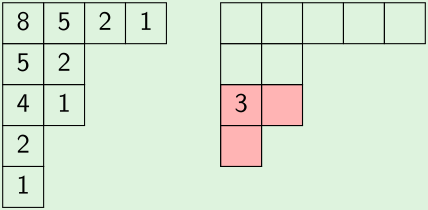
p-Core Partitions

Definition

A partition is *p-core* if none of its hook lengths are divisible by p.

Example

The first partition is 3-core, while the second is not:



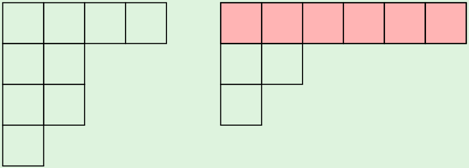
p'-Partitions

Definition

A partition is a p'-partition if none of its parts are divisible by p.

Example

The partition $(4, 2, 2, 1)$ is a $3'$-partition; while $(6, 2, 1)$ is not:



Maximal p-Core p'-Partitions

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Theorem (McDowell, McSpirit–Ono)

Any p-core p'-partition λ must satisfy

$$|\lambda| \leq \frac{1}{24}(p^6 - 4p^5 + 5p^4 + 12p^3 - 42p^2 + 52p - 24).$$

On the other hand, there exists a p-core p'-partition with

$$|\lambda| = \frac{1}{96}(p^6 + 6p^4 - 12p^3 + 89p^2 - 120p - 48).$$

Maximal p-Core p'-Partitions

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Let Λ_p denote the **unique** maximal p-core p'-partition.

Question

How does $|\Lambda_p|$ behave as $p \to \infty$?

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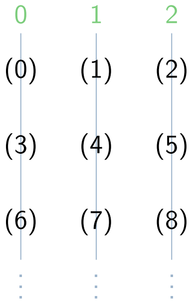
Theorem (D)

For all $p > 10^6$, we have

$$\frac{1}{24}p^6 - p^5\sqrt{p} < |\Lambda_p| < \frac{1}{24}p^6 - \frac{1}{200}p^5\sqrt{p}.$$

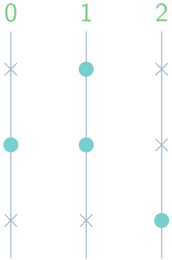
Abacus Notation

The p-abacus consists of p vertical runners, labelled 0 through $p-1$, with positions read from left to right and top to bottom.



Abacus Notation

Positions are either *beads* or *gaps*; position 0 is required to be a gap.



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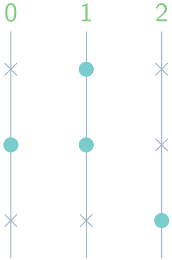


Each bead contributes a part equal to the number of preceding gaps.

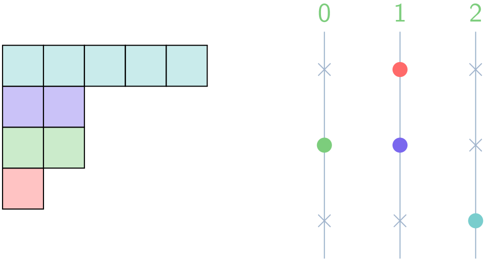
Example

The above abacus corresponds to the partition $(5, 2, 2, 1)$.

Abacus Notation

Fact

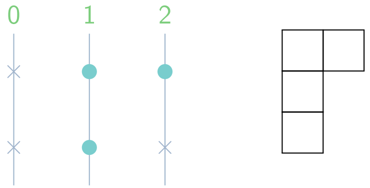
Every partition corresponds to a unique abacus.



p-Core Partitions in Abacus Notation

Fact

A partition is p-core if and only if in its abacus notation, all beads are topmost in their runners.



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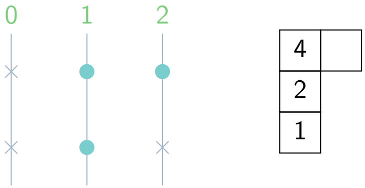

Proof outline.

The hook lengths in the leftmost column mod p correspond to the runner labels of their beads, so there are no beads on runner 0.

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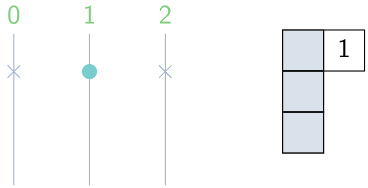


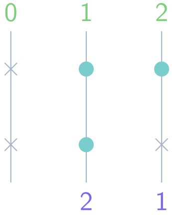
Proof outline.

The hook lengths in the leftmost column mod p correspond to the runner labels of their beads, so there are no beads on runner 0. Then delete the first column by deleting everything before the second gap (moving it to position 0), so there are no beads below the second gap. And so on. $\square$

Bead Multiplicities

Definition

The ith *bead multiplicity* b_i is the number of beads on runner i.



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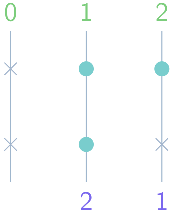


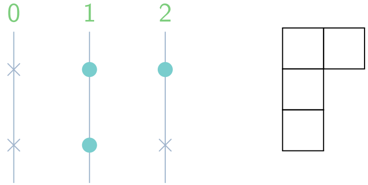
Lemma

$$|\lambda| = -\frac{1}{2}\left(\sum_{i=1}^{p-1} b_i\right)^2 + \frac{p}{2}\sum_{i=1}^{p-1} b_i^2 + \sum_{i=1}^{p-1}\left(i - \frac{p-1}{2}\right) b_i.$$

More About Abacus Notation

Fact

A maximal p-core p′-partition has all beads rightmost in their rows.



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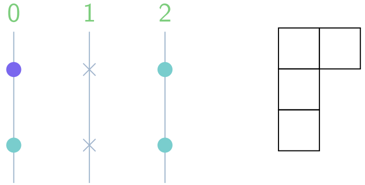

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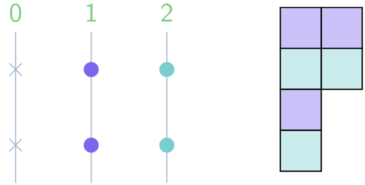

Proof Outline.

First, add beads to the start if necessary, so that the last runner has the most beads. Then shift all beads to the right end of their row. □

p'-Partitions in Abacus Notation

Definition

Call an abacus *aligned* if it has both properties.

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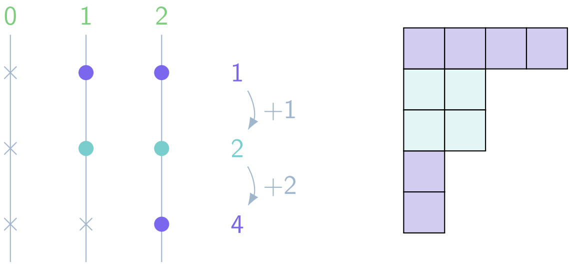
Fact

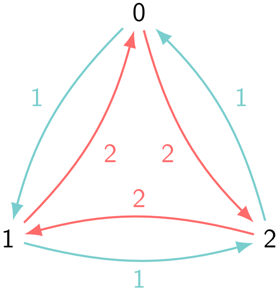
For an aligned abacus, all beads in a row contribute equal parts; if the row contains i gaps followed by $p - i$ beads, these parts are i more than the parts corresponding to the previous row.



The Additive Residue Graph

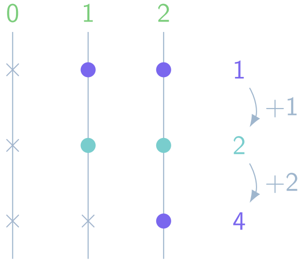
Definition

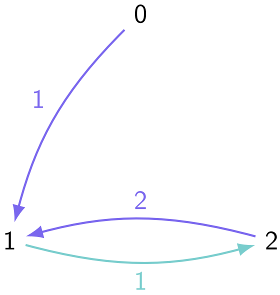
The *additive residue graph* $\mathcal{G}_p$ has vertices for the residues mod p, and edges $x \to x + i$ labelled i, for every residue x and every $1 \leq i \leq p - 1$.



Walks on $\mathcal{G}_p$

Aligned abaci correspond to walks on $\mathcal{G}_p$: start at 0, and for a row with i gaps, take the edge labelled i. This walk has nondecreasing edge labels; any such walk corresponds to an aligned abacus.





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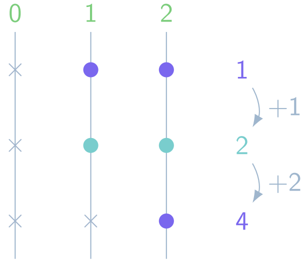

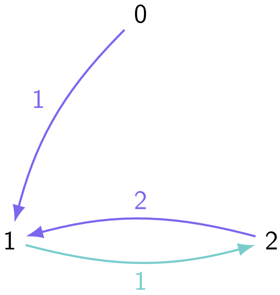

Fact

The abacus corresponds to a p'-partition iff the walk never returns to 0.

Long Walks and Λ_p

Theorem (McDowell)

The unique maximal p-core p'-partition Λ_p corresponds to the longest valid walk on $\mathcal{G}_p$.

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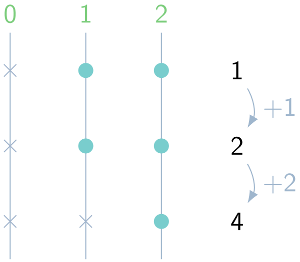
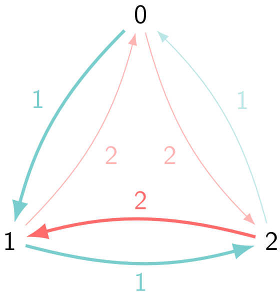




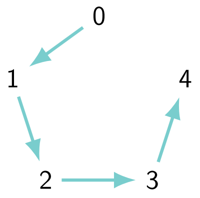
The Longest Walk

Theorem (McDowell)

The longest walk on $\mathcal{G}_p$ has an i-edge incident to $p-1$ for every i.

This means the longest walk can be split into "independent" segments:

- Start at 0 and take $(p-1)$ 1-steps to $p-1$.



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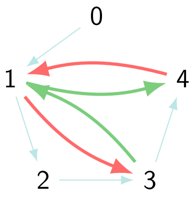

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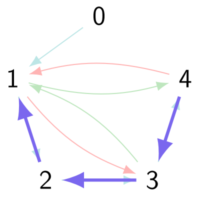
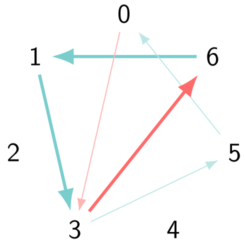
- Start at $p-1$, and take $(p-2)$ $(p-1)$-steps to 1.



Analyzing the Segments

Think of the ith segment as walking from $p-1$ all the way to 0 (using i-edges) and back to $p-1$ (using $(i+1)$-edges), and then cutting off a loop around 0.



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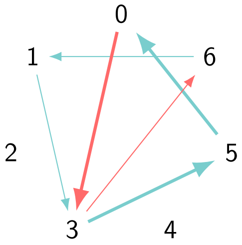


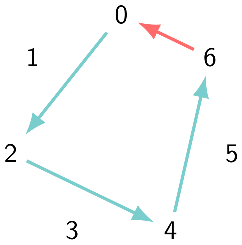
Focus on the part we're **cutting off** — say the entire loop (going all the way to 0) contains $x_i^{\max}$ i-edges and $y_i^{\max}$ $(i+1)$-edges, and the part cut off contains x_i i-edges and y_i $(i+1)$-edges.

The Subtractions

Fact

If we didn't cut off anything, the total number of i-edges would be p.

In other words, $y_{i-1}^{\max} + x_i^{\max} = p$.



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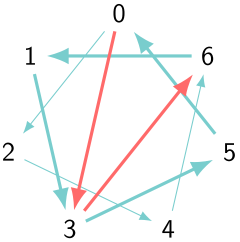

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On average, the subtractions x_i and y_i are small (on the order of $\sqrt{p}$).

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Our proof will proceed in several steps:

- Find a way to estimate the x_i and y_i.
- Find upper and lower bounds on $\sum(x_i + y_i)$ which are on the order of $p\sqrt{p}$.
- Use the formula for the size of a partition given its bead multiplicities, and translate these results to bounds on $|\Lambda_p|$.

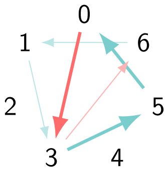
Equations for x_i and y_i

Lemma

(x_i, y_i) is the solution with minimal $x + y$ to

$$ix + (i+1)y \equiv 0 \pmod{p}$$

where $0 < x \leq x_i^{\max}$ and $0 < y \leq y_i^{\max}$.



Example

The minimal solution to $2x + 3y \equiv 0 \pmod{7}$ with $0 < x \leq 4$ and $0 < y \leq 2$ is $(2, 1)$.

Finding a Nicer Equation

Lemma

Every $1 \leq i \leq p-2$ *can be written as*

$$\frac{i+1}{i} \equiv -\frac{r}{s} \text{ or } \frac{r}{s} \pmod{p},$$

for relatively prime $0 < r, s < \sqrt{p}$.

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Lemma

In each case, the pair (r, s) is unique — there is at most one way to write $\frac{i+1}{i} \equiv -\frac{r}{s}$, and at most one way to write $\frac{i+1}{i} \equiv \frac{r}{s}$.

The First Case

If we can write $\frac{i+1}{i} \equiv -\frac{r}{s} \pmod{p}$, then our equation for (x_i, y_i) becomes

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Proof.

Note that (r, s) is a solution to $sx - ry \equiv 0 \pmod{p}$.
It remains to check that $r \leq x_i^{\max}$ and $s \leq y_i^{\max}$. We can do this by explicitly computing

$$x_i^{\max} \equiv \frac{1}{i} \equiv -\frac{r+s}{s} \pmod{p} \implies sx_i^{\max} + r + s \geq p.$$

□

The Second Case

Meanwhile, if $\frac{i+1}{i} \equiv \frac{r}{s} \pmod{p}$, the equation becomes

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Proof of Lower Bound.

For any solution (x, y),

$$\max(r,s) \cdot (x + y) > sx + ry \geq p.$$

□

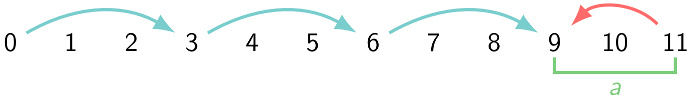
The Second Case

Proof of Upper Bound.

WLOG $s > r$. Choose $0 < a < rs$ with $s \mid (p - a)$ and $r \mid a$, and take

$$(x, y) = \left(\frac{p-a}{s}, \frac{a}{r}\right).$$

We can check $x \leq x_i^{\max}$ and $y \leq y_i^{\max}$ as in the previous case. $\square$



Upper Bound on Subtractions

Lemma

$$\sum_{i=1}^{p-2}(x_i + y_i) < \frac{11}{3}p\sqrt{p}.$$

Upper Bound on Subtractions

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$$\sum_{i=1}^{p-2}(x_i + y_i) < \frac{11}{3} p\sqrt{p}.$$

Proof.

For the $\frac{i+1}{i} \equiv -\frac{r}{s}$ case, the total contribution is at most

$$\sum_{(r,s)}(r+s) < 2(\sqrt{p}-1)\sum_{r<\sqrt{p}} r < p\sqrt{p}.$$

Upper Bound on Subtractions

Proof (Cont.)

For the $\frac{i+1}{i} \equiv \frac{r}{s}$ case, the total contribution is less than

$$\sum_{(r,s)} \frac{p}{\max(r,s)} + \max(r,s) - 1.$$

Upper Bound on Subtractions

Proof (Cont.)

For the $\frac{i+1}{i} \equiv \frac{r}{s}$ case, the total contribution is less than

$$\sum_{(r,s)} \frac{p}{\max(r,s)} + \max(r,s) - 1.$$

Every $m = \max(r,s)$ occurs less than $2m$ times, giving the upper bound

$$\sum_{m<\sqrt{p}} 2m\left(\frac{p}{m} + m - 1\right) < \frac{8}{3}p\sqrt{p}.$$

□

Lower Bound on Subtractions

Lemma

$$\sum_{i=1}^{p-2}(x_i + y_i) > \frac{6}{5}p\sqrt{p} - 16p.$$

Lower Bound on Subtractions

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$$\sum_{i=1}^{p-2}(x_i + y_i) > \frac{6}{5}p\sqrt{p} - 16p.$$

Proof.

Only consider the $\frac{i+1}{i} \equiv \frac{r}{s}$ case. Every $m = \max(r, s)$ occurs exactly $2\varphi(m)$ times (if $m > 1$), giving the lower bound

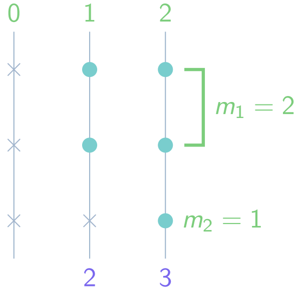
$$\sum_{2 \leq m < \sqrt{p}} 2\varphi(m) \cdot \frac{p}{m} \approx 2p \cdot \frac{6}{\pi^2}\sqrt{p}.$$

□

Row Multiplicities

Definition

The ith *row multiplicity* m_i is the number of rows with i gaps.

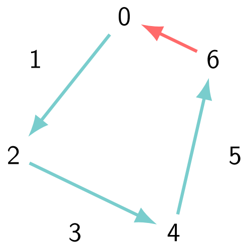


Then $b_i = m_1 + m_2 + \cdots + m_i$.

Row Multiplicities

Fact

For all $2 \leq i \leq p-2$, $m_i = p - y_{i-1} - x_i$.



- For the $(i-1)$th segment, we take $y_{i-1}^{\max}$ i-edges, and cut off y_{i-1}.

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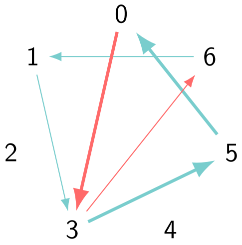

- For the $(i-1)$th segment, we take $y_{i-1}^{\max}$ i-edges, and cut off y_{i-1}.
- For the ith segment, we take $x_i^{\max}$ i-edges, and cut off x_i.

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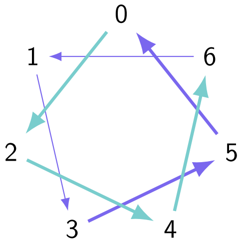
For all $2 \leq i \leq p-2$, $m_i = p - y_{i-1} - x_i$.

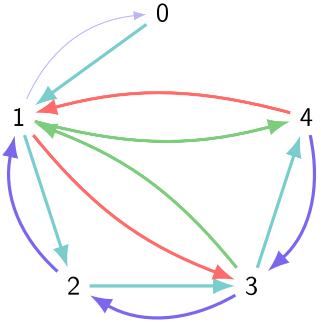


- For the $(i-1)$th segment, we take $y_{i-1}^{\max}$ i-edges, and cut off y_{i-1}.
- For the ith segment, we take $x_i^{\max}$ i-edges, and cut off x_i.
- We know $y_{i-1}^{\max} + x_i^{\max} = p$.

A Useful Symmetry

Fact

For all $2 \leq i \leq p-2$, $m_i = m_{p-i}$ *(while* $m_{p-1} = m_1 - 1$*).*



Cumulative Subtractions

We have

$$b_i = (p - x_1) + (p - y_1 - x_2) + \cdots + (p - y_{i-1} + x_i),$$

so define $c_i = ip - b_i$ and $c = \sum_{i=1}^{p-2}(x_i + y_i)$.

Cumulative Subtractions

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Fact

We have $c_i + c_{p-1-i} = c$ for all $1 \leq i \leq p-2$, and $c_{p-1} = c + 1$.

The Theorem

Theorem (D)

For all $p > 10^6$, we have

$$\frac{1}{24}p^6 - p^5\sqrt{p} < |\Lambda_p| < \frac{1}{24}p^6 - \frac{1}{200}p^5\sqrt{p}.$$

The Theorem

Theorem (D)

For all $p > 10^6$, we have

$$\frac{1}{24}p^6 - p^5\sqrt{p} < |\Lambda_p| < \frac{1}{24}p^6 - \frac{1}{200}p^5\sqrt{p}.$$

Recall that

$$|\Lambda_p| = -\frac{1}{2}\left(\sum_{i=1}^{p-1} b_i\right)^2 + \frac{p}{2}\sum_{i=1}^{p-1} b_i^2 + \sum_{i=1}^{p-1}\left(i - \frac{p-1}{2}\right) b_i,$$

where $b_i = ip - c_i$. The idea is to translate our previous bounds on c to bounds on $|\Lambda_p|$, using this formula.

The Lower Bound

Proof of Lower Bound.

Using the symmetry $c_i + c_{p-1-i} = c$, we get

$$\sum_{i=1}^{p-1} b_i = \sum_{i=1}^{p-1} (ip - c_i) \approx \frac{p^3 - pc}{2}.$$

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For the second term, use the bound

$$\sum_{i=1}^{p-1} b_i^2 > \sum_{i=1}^{p-1} (i^2 p^2 - 2ipc) \approx \frac{p^5}{3} - p^3 c.$$

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Combining these and using $c < \frac{11}{3} p\sqrt{p}$ gives a lower bound around

$$-\frac{1}{2}\left(\frac{p^3 - pc}{2}\right)^2 + \frac{p}{2}\left(\frac{p^5}{3} - p^3 c\right) \approx \frac{p^6}{24} - \frac{p^4 c}{4} > \frac{p^6}{24} - p^5\sqrt{p}. \quad \square$$

The Upper Bound

Proof of Upper Bound.

Again, the first term is around

$$-\frac{1}{2}\left(\sum_{i=1}^{p-1} b_i\right) \approx -\frac{p^6}{8} + \frac{p^4 c}{4}.$$

The Upper Bound

Proof of Upper Bound.

Again, the first term is around

$$-\frac{1}{2}\left(\sum_{i=1}^{p-1} b_i\right) \approx -\frac{p^6}{8} + \frac{p^4 c}{4}.$$

For the second, pair terms and use symmetry: $b_i^2 + b_{p-1-i}^2$ is around

$$p^2(i^2 + (p-1-i)^2) - cp(p-1) - p(c - 2c_i)(p-1-2i).$$

Another Lemma

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$$c_{\lfloor p/18 \rfloor} < \frac{2}{5} p\sqrt{p} + p.$$

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Proof.

The largest bounds are $\frac{p}{m} + m - 1$, where $\frac{i+1}{i} \equiv \frac{r}{s}$ with $\max(r, s) = m$.

Another Lemma

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$$c_{\lfloor p/18 \rfloor} < \frac{2}{5} p\sqrt{p} + p.$$

Proof.

The largest bounds are $\frac{p}{m} + m - 1$, where $\frac{i+1}{i} \equiv \frac{r}{s}$ with $\max(r, s) = m$. Each m occurs for at most m pairs (r, s). Since

$$1 + 2 + \cdots + \frac{\sqrt{p}}{3} \approx \frac{p}{18},$$

by bounding each term individually we get

$$\sum_{m < \sqrt{p}/3+1} m\left(\frac{p}{m} + m - 1\right) < \frac{2}{5} p\sqrt{p} + p.$$

□

Finishing the Upper Bound

Proof of Upper Bound (Cont.)

Recall that $b_i^2 + b_{p-1-i}^2$ was around

$$p^2(i^2 + (p-1-i)^2) - cp(p-1) - p(c - 2c_i)(p-1-2i).$$

Finishing the Upper Bound

Proof of Upper Bound (Cont.)

Recall that $b_i^2 + b_{p-1-i}^2$ was around

$$p^2(i^2 + (p-1-i)^2) - cp(p-1) - p(c-2c_i)(p-1-2i).$$

Now, for $i < \frac{p}{18}$, the last term has nontrivial contribution! Use

$$c - 2c_i > \frac{6}{5}p\sqrt{p} - 16p - 2\left(\frac{2}{5}p\sqrt{p} + p\right),$$

and $p - 1 - 2i > \frac{8}{9}p - 1$.

Finishing the Upper Bound

Proof of Upper Bound (Cont.)

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and $p - 1 - 2i > \frac{8}{9}p - 1$.
Combining everything gets a bound of around

$$\left(-\frac{p^6}{8} + \frac{p^4c}{4}\right) + \left(\frac{p^6}{6} - \frac{p^4c}{4} - \frac{p^5\sqrt{p}}{120}\right).$$

□

Summary

Theorem (D)

For all $p > 10^6$, we have

$$\frac{1}{24}p^6 - p^5\sqrt{p} < |\Lambda_p| < \frac{1}{24}p^6 - \frac{1}{200}p^5\sqrt{p}.$$

Summary

Theorem (D)

For all $p > 10^6$, we have

$$\frac{1}{24}p^6 - p^5\sqrt{p} < |\Lambda_p| < \frac{1}{24}p^6 - \frac{1}{200}p^5\sqrt{p}.$$

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