A local properties problem for difference sets

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Suppose we have a big set of numbers, and we know that every small subset has lots of distinct differences. How many distinct differences does this force the *entire* set to have?

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Observation

Any n-element set A satisfies $n - 1 \leq |A - A| \leq \binom{n}{2}$. So we consider ℓ with $k - 1 \leq \ell \leq \binom{k}{2}$; then $g(n, k, \ell)$ is always at least linear in n, and at most quadratic in n.

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Theorem (Li '22)

For each k, the superlinear threshold is $k-1$.

Theorem (Li '22)

For each k, the quadratic threshold is at most $\approx \frac{3}{8}k^2$.

Previous bounds

Several lower bounds are known.

- Fish, Pohoata, Sheffer (2020) proved a family of lower bounds for ℓ between $\approx \frac{7}{32}k^2$ and $\approx \frac{1}{4}k^2$ — e.g., when $4 \mid k$, we have

$$g\left(n, k, \frac{k^2}{4} + 1\right) = \Omega(n^{2-\frac{8}{k}}).$$

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Some upper bounds are known for ‘small’ ℓ (compared to k^2), due to Fish–Lund–Sheffer (2019), Fish–Pohoata–Sheffer (2020), and Li (2022).

- Fish, Lund, Sheffer (2019) proved that

$$g\left(n, k, \frac{k^{\log_2 3} - 1}{2}\right) = O(n^{\log_2 3}).$$

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- *For k odd, the quadratic threshold is between $\frac{(k+1)^2}{4}-3$ and $\frac{(k+1)^2}{4}$.*

- To prove that $g(n,k,\frac{k^2}{4}+1)=\Omega(n^2)$, we show that any set A with $|A-A|\ll n^2$ must contain k elements with

$$a_1+a_2=a_3+a_4=\cdots=a_{k-1}+a_k.$$

Then $\{a_1,\ldots,a_k\}$ has at most $\frac{k^2}{4}$ distinct differences.

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- To prove that $g(n, k, \frac{k^2}{4} + 1) = \Omega(n^2)$, we show that any set A with $|A - A| \ll n^2$ must contain k elements with

$$a_1 + a_2 = a_3 + a_4 = \cdots = a_{k-1} + a_k.$$

  Then $\{a_1, \ldots, a_k\}$ has at most $\frac{k^2}{4}$ distinct differences.
- To prove that $g(n, k, \frac{k^2}{4}) = o(n^2)$, we use a random construction. We analyze which k-element 'configurations' are expected to appear in it, and show that all of them have at least $\frac{k^2}{4}$ distinct differences (i.e., the configuration $a_1 + a_2 = \cdots = a_{k-1} + a_k$ is the 'worst').

Intermediate bounds

Theorem (D. '23+)

For all $1 < c \leq 2$, *we have* $g(n,k,\ell) = o(n^c)$ *for* $\ell \approx \left(\frac{c-1}{c}\right)^2 k^2$.

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For all $t \in \mathbb{N}$*, we have* $g(n,k,\ell) = \Omega(n^{1+\frac{1}{2^t-1}})$ *for* $\ell \approx \frac{1}{3} \cdot \left(\frac{3}{4}\right)^t k^2$.

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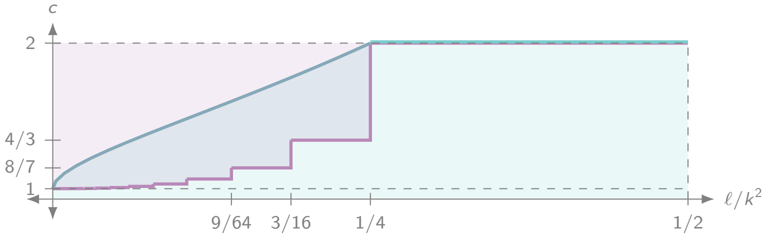

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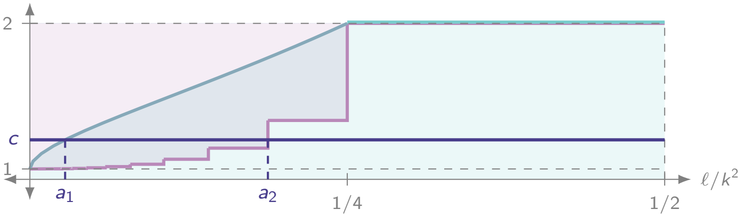

The number of possible exponents

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We define S_k as the set of 'possible exponents' of n in $g(n,k,\ell)$, i.e.,

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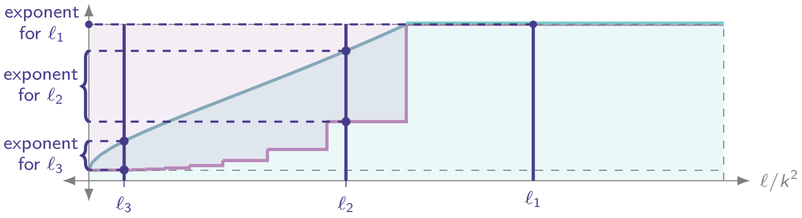

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Thanks for listening!