Distinct distances between a line and strip

Sanjana Das

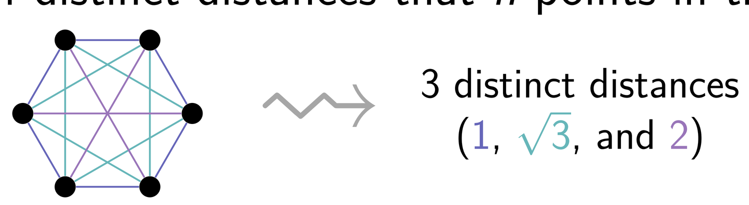
Mentor: Adam Sheffer

NYC Discrete Math REU

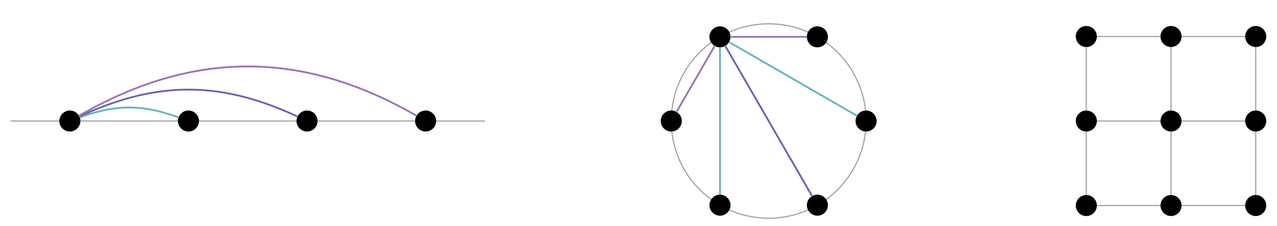
Background

The distinct distances problem

The original distinct distances problem, asked by Erdős (1946): What is the minimum number of distinct distances that n points in the plane can form?



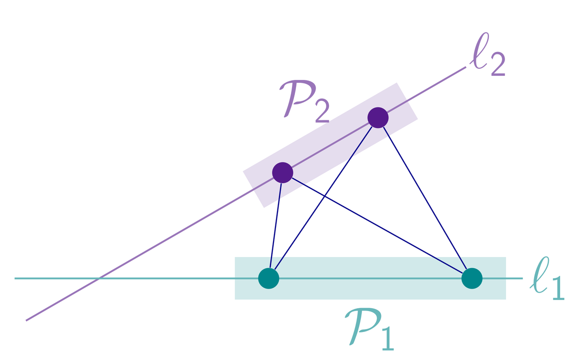
For example, n equally spaced points on a line form $n-1$ distances, and n equally spaced points on a circle form $\lfloor n/2 \rfloor$. Erdős found a slightly better construction — a $\sqrt{n} \times \sqrt{n}$ lattice forms $O(n/\sqrt{\log n})$ distances.



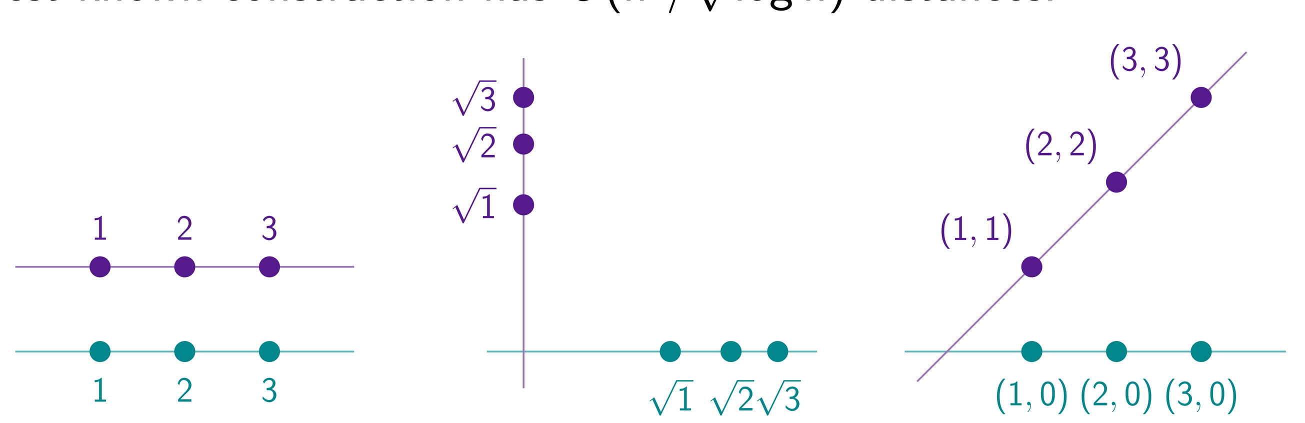
Many people have worked on lower bounds, starting with Erdős's bound that any n points form $\Omega(n^{1/2})$ distances. Finally, Guth and Katz (2010) proved a bound of $\Omega(n/\log n)$, almost matching the upper bound.

Distinct distances between two lines

Although the original distinct distances problem is nearly settled, most of its variants are far from being solved. In one of the simplest-looking variants, we have two lines ℓ_1 and ℓ_2, a set $\mathcal{P}_1$ of n points on ℓ_1, and another set $\mathcal{P}_2$ of n points on ℓ_2. We're interested in the minimum possible number of distances between $\mathcal{P}_1$ and $\mathcal{P}_2$.



If ℓ_1 and ℓ_2 are parallel or perpendicular, there are constructions with $O(n)$ distances, and this is the minimum possible. So we assume they're not; then the best known construction has $O(n^2/\sqrt{\log n})$ distances.

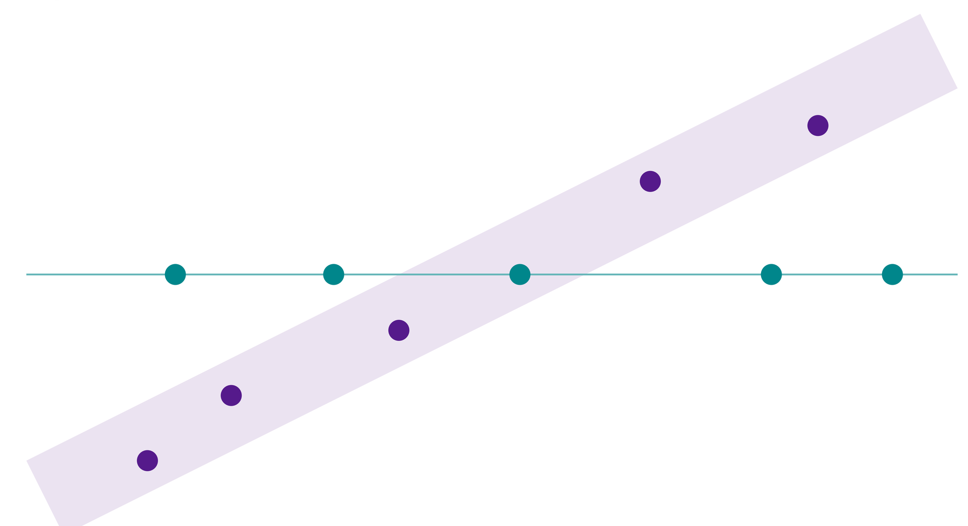


the parallel case $O(n)$ distances | the perpendicular case $O(n)$ distances | the general case $O(n^2/\sqrt{\log n})$ distances

Meanwhile the current best lower bound (when ℓ_1 and ℓ_2 are not parallel or perpendicular) is $\Omega(n^{3/2})$, due to Solymosi and Zahl (2024).

Our work

We study a generalization of distinct distances between two lines, where $\mathcal{P}_1$ still lies on a line, but $\mathcal{P}_2$ lies on a *strip* around a line. We're again interested in the minimum possible number of distances between $\mathcal{P}_1$ and $\mathcal{P}_2$.

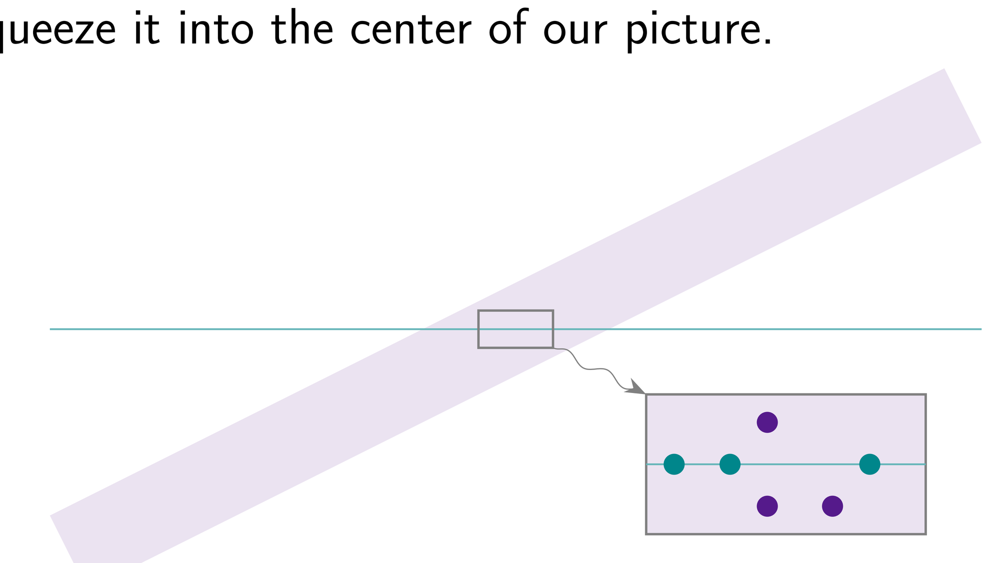


Theorem (D.–S.)

Consider a set $\mathcal{P}_1$ of n points on the line $y=0$ and a set $\mathcal{P}_2$ of n points on the strip $|y - sx| \le \varepsilon$. Suppose that in each set, the x-coordinates of every two points differ by at least $32\varepsilon/s$. Then

$$\#(\text{distinct distances}) \ge n^{22/15 - o(1)}.$$

The condition that the points are reasonably spaced out is necessary for the statement that $\mathcal{P}_2$ lies on a strip to be meaningful. Otherwise, we could take any configuration with $\mathcal{P}_1$ on a line but with no constraints on $\mathcal{P}_2$, shrink it down, and squeeze it into the center of our picture.



References

[BS18] Ariel Bruner and Micha Sharir.
Distinct distances between a collinear set and an arbitrary set of points.
Discrete Mathematics, 2018.

[SSS13] Micha Sharir, Adam Sheffer, and József Solymosi.
Distinct distances on two lines.
Journal of Combinatorial Theory, Series A, 2013.

[SZ24] József Solymosi and Joshua Zahl.
Improved Elekes–Szabó type estimates using proximity.
Journal of Combinatorial Theory, Series A, 2024.

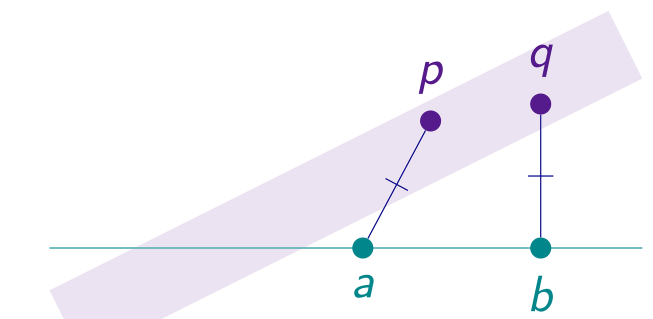
Proof ideas

Distance energy and incidences

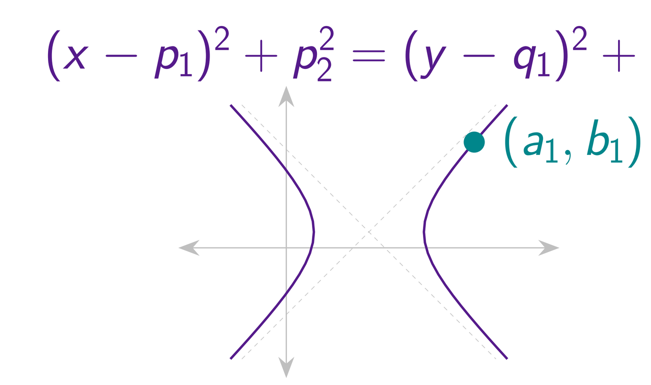
We consider the distance energy

$$E(\mathcal{P}_1, \mathcal{P}_2) = \#\{(a, p, b, q) \in (\mathcal{P}_1 \times \mathcal{P}_2)^2 \mid |ap| = |bq|\}.$$

Few distinct distances implies a large distance energy, so to lower-bound the number of distinct distances, it suffices to *upper*-bound $E(\mathcal{P}_1, \mathcal{P}_2)$.



We do this using incidence bounds — we turn each pair (a, b) into a point in $\mathbb{R}^2$, and each pair (p, q) into a curve expressing the equation $|ap| = |bq|$.



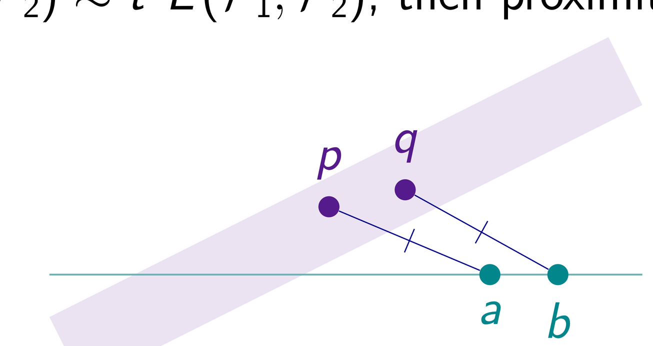
This produces sets $\mathcal{P}$ and $\mathcal{H}$ of n^2 points and hyperbolas such that the distance energy counts their incidences. Then an incidence bound gives

$$E(\mathcal{P}_1, \mathcal{P}_2) = I(\mathcal{P}, \mathcal{H}) \lesssim |\mathcal{P}|^{6/11} |\mathcal{H}|^{9/11} = n^{30/11}.$$

Bruner and Sharir (2018) used this method to show the number of distances between a line and a (mostly) unrestricted set is at least $n^{14/11 - o(1)}$.

Proximity

To improve the above bound, we use proximity, introduced in [SZ24] — we define $E_t(\mathcal{P}_1, \mathcal{P}_2)$ to only count quadruples where b is one of the tn closest points to a, and q to p. Intuitively, if $|ap| = |bq|$ and p and q are close, then so are a and b; this lets us show $E_t(\mathcal{P}_1, \mathcal{P}_2) \gtrsim tE(\mathcal{P}_1, \mathcal{P}_2)$. (Otherwise, we might expect $E_t(\mathcal{P}_1, \mathcal{P}_2) \approx t^2 E(\mathcal{P}_1, \mathcal{P}_2)$; then proximity wouldn't help.)



Meanwhile, the new incidence problem has tn^2 points and hyperbolas, so the same incidence bound gives

$$E_t(\mathcal{P}_1, \mathcal{P}_2) = I(\mathcal{P}_t, \mathcal{H}_t) \lesssim |\mathcal{P}_t|^{6/11} |\mathcal{H}_t|^{9/11} = t^{15/11} n^{30/11}.$$

So we get $E_t(\mathcal{P}_1, \mathcal{P}_2) \lesssim t^{4/11} n^{30/11}$, and taking t small gives a better bound.