Distinct distances between a line and strip

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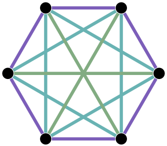
The distinct distances problem

Imagine we've got a set of points $\mathcal{P} \subseteq \mathbb{R}^2$. We're interested in the number of distinct distances between two points in $\mathcal{P}$.



$\rightsquigarrow$

3 distinct distances
(1, $\sqrt{3}$, and 2)

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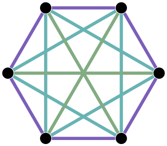


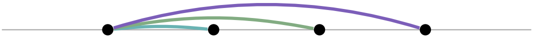
$\rightsquigarrow$ 3 distinct distances (1, $\sqrt{3}$, and 2)

Question (Erdős 1946)

What's the minimum number of distinct distances in a set of n points?

Upper bounds

- n equally spaced points on a line have $n - 1$ distinct distances.



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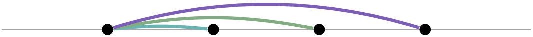
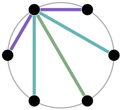


- n equally spaced points on a circle have $\lfloor n/2 \rfloor$ distinct distances.



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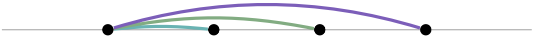

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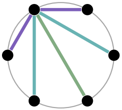
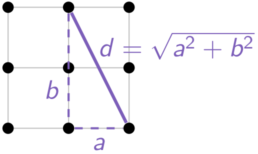


- (Erdős 1946) A $\sqrt{n} \times \sqrt{n}$ lattice has $O(n/\sqrt{\log n})$ distinct distances.



The idea is that $a, b \leq \sqrt{n}$, so $a^2 + b^2 \in \{1, \ldots, 2n\}$; and only a $1/\sqrt{\log n}$ fraction of integers in this range are a sum of squares.

Lower bounds

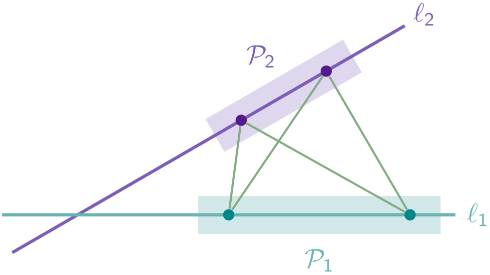
Lower bound	Authors
$\Omega(n^{1/2})$	Erdős 1946
$\Omega(n^{2/3})$	Moser 1952
$\Omega(n^{5/7})$	Chung 1984
$\Omega(n^{4/5}/\log n)$	Chung–Szemerédi–Trotter 1992
$\Omega(n^{4/5})$	Székely 1997
$\Omega(n^{6/7})$	Solymosi–Tóth 2001
$\Omega(n^{0.8634})$	Tardos 2001
$\Omega(n^{0.8641})$	Katz–Tardos 2004
$\Omega(n/\log n)$	Guth–Katz 2010

The Guth–Katz result solves the problem up to a factor of $\sqrt{\log n}$.

A distinct distances variant

Question

Suppose we have two lines ℓ_1 and ℓ_2, and two sets of points $\mathcal{P}_1 \subseteq \ell_1$ and $\mathcal{P}_2 \subseteq \ell_2$ (with n points each). What's the minimum number of distinct distances between $\mathcal{P}_1$ and $\mathcal{P}_2$?



A simple lower bound

Claim

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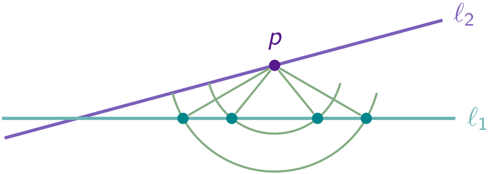
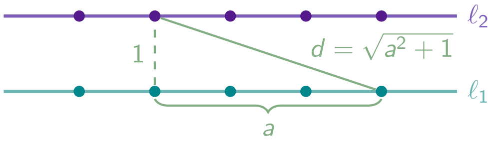


Proof.

- Fix $p \in \mathcal{P}_2$; we'll consider only distances between $\mathcal{P}_1$ and p.
- Each distance from p is repeated at most twice.
- There are n points in $\mathcal{P}_1$, so at least $n/2$ distinct distances. □

Upper bounds

- If $\ell_1 \parallel \ell_2$, there are constructions with $O(n)$ distinct distances.



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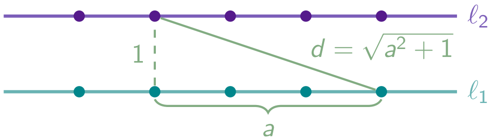
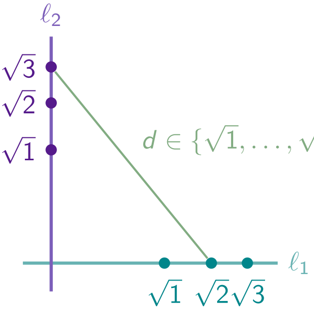
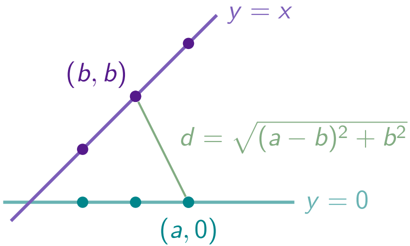


- The same is true if $\ell_1 \perp \ell_2$.



Upper bounds

Otherwise, the best construction we know of has $O(n^2/\sqrt{\log n})$ distinct distances (the factor of $\sqrt{\log n}$ comes from the same number-theoretic fact about sums of squares).



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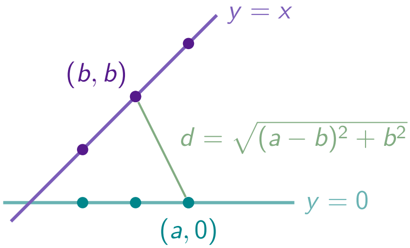


Question

If we assume ℓ_1 and ℓ_2 are not parallel or perpendicular, can we get a better lower bound than $\Omega(n)$?

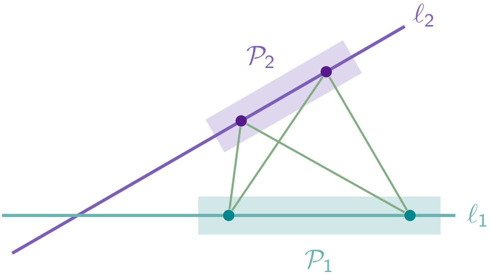
Better lower bounds

Theorem (Sharir–Sheffer–Solymosi 2013)

$\#(\text{distinct distances}) = \Omega(n^{4/3})$.

Theorem (Solymosi–Zahl 2024)

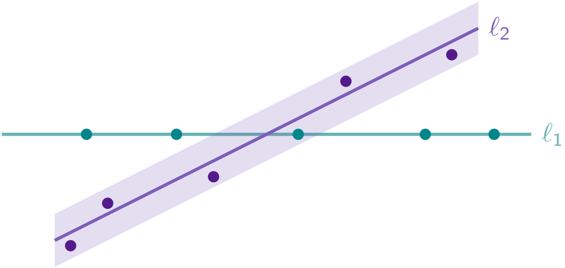
$\#(\text{distinct distances}) = \Omega(n^{3/2})$.



Distances between a line and strip

Question

What if instead of lying on a line, $\mathcal{P}_2$ lies on a strip?



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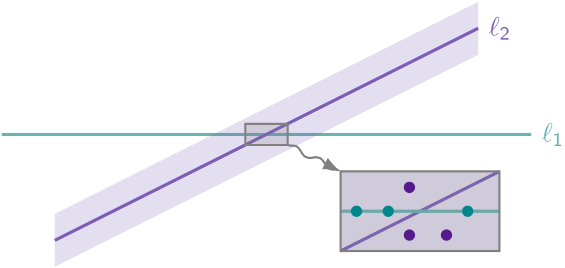


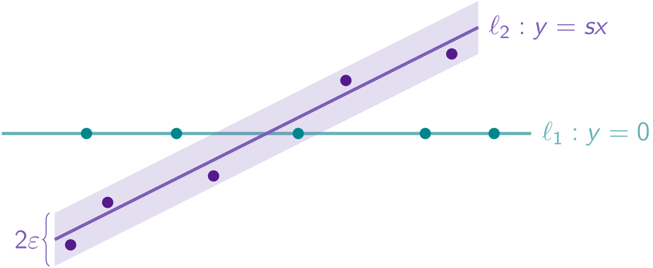
For this to be meaningful, we need to put a spacing condition on the points — otherwise we could squeeze a configuration with arbitrary $\mathcal{P}_2$ into the center of the picture.

Our result

Theorem (D.–Sheffer 2024++)

Suppose that $\mathcal{P}_1$ lies on the line $y = 0$ and $\mathcal{P}_2$ on the strip $|y - sx| \leq \varepsilon$, and both have x-coordinates spaced out by at least ε/s. Then

$$\#(\text{distinct distances}) \gtrsim n^{22/15-o(1)} \approx n^{1.46}.$$



SSS13 — distance energy

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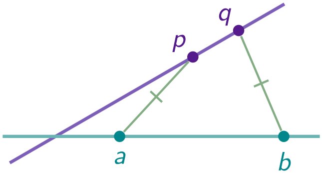
The idea of the proof is to consider the distance energy

$$E(\mathcal{P}_1, \mathcal{P}_2) = \#\{(a, p, b, q) \in (\mathcal{P}_1 \times \mathcal{P}_2)^2 \mid |ap| = |bq|\}.$$



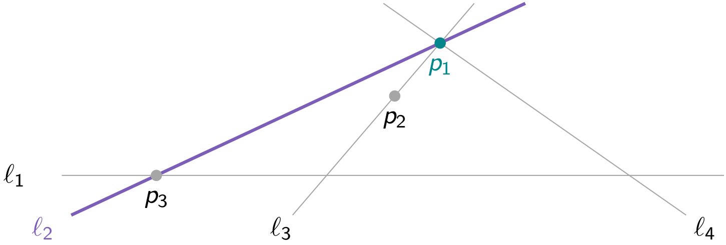
If the number of distinct distances is small, then $E(\mathcal{P}_1, \mathcal{P}_2)$ is large — intuitively, few possible values of $|ap|$ means lots of collisions.

SSS13 — incidences

Definition

Given a set of points $\mathcal{P}$ and curves $\mathcal{C}$, their number of incidences is

$$I(\mathcal{P}, \mathcal{C}) = \#\{(p, c) \in \mathcal{P} \times \mathcal{C} \mid \text{point } p \text{ lies on curve } c\}.$$



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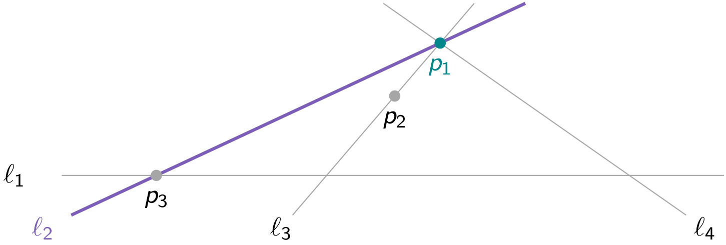

There are tools for upper-bounding $I(\mathcal{P},\mathcal{C})$ under certain conditions on $\mathcal{P}$ and $\mathcal{C}$ ('incidence bounds').

SSS13 — from distance energy to incidences

We want to upper-bound $E(\mathcal{P}_1, \mathcal{P}_2) = \#\{(a, p, b, q) \mid |ap| = |bq|\}$.

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- Let ℓ_1 be the x-axis and ℓ_2 the line $y = sx$, so that $a = (a_1, 0)$, $p = (p_1, sp_1)$, and so on. Then $|ap| = |bq|$ means

$$(a_1 - p_1)^2 + (sp_1)^2 = (b_1 - q_1)^2 + (sq_1)^2.$$

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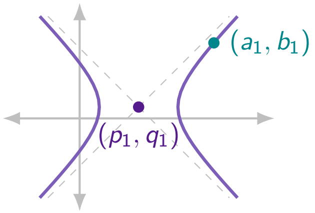
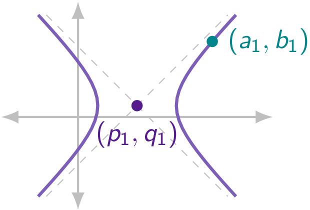
- We can turn this into an incidence problem in $\mathbb{R}^2$ by letting a and b define a point, and p and q a hyperbola:
  - $\mathcal{P} = \{(a_1, b_1) \mid a, b \in \mathcal{P}_1\}$.
  - $\mathcal{H} = \{(x - p_1)^2 + (sp_1)^2 = (y - q_1)^2 + (sq_1)^2 \mid p, q \in \mathcal{P}_2\}$.



SSS13 — applying an incidence bound

We've made sets of n^2 points $\mathcal{P}$ (one for each $a, b \in \mathcal{P}_1$) and hyperbolas $\mathcal{H}$ (one for each $p, q \in \mathcal{P}_2$) such that $|ap| = |bq|$ means the point defined by (a, b) lies on the hyperbola defined by (p, q).



Then $E(\mathcal{P}_1, \mathcal{P}_2)$ is the number of incidences between these points and hyperbolas, and an incidence bound gives

$$E(\mathcal{P}_1, \mathcal{P}_2) = I(\mathcal{P}, \mathcal{H}) \lesssim |\mathcal{P}|^{2/3} |\mathcal{H}|^{2/3} = n^{8/3}.$$

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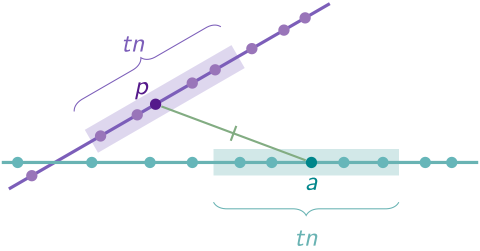
$$E(\mathcal{P}_1, \mathcal{P}_2) = \{(a, p, b, q) \in (\mathcal{P}_1 \times \mathcal{P}_2)^2 \mid |ap| = |bq|\}.$$

Now we consider the t-proximal distance energy $E_t(\mathcal{P}_1, \mathcal{P}_2)$ (for some $t \in (0, 1])$, where we also require that b is one of the tn closest points to a, and q is one of the tn closest points to p.



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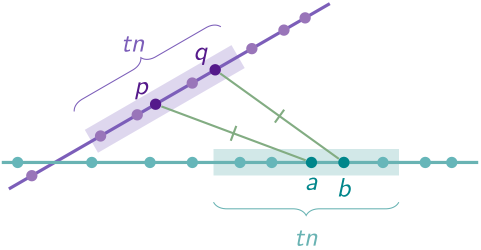

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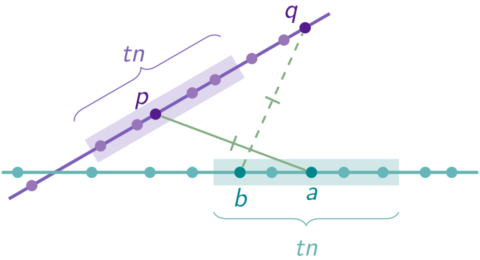
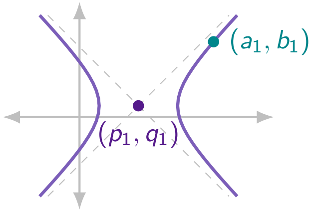


SZ22 — intuition behind proximity

- We only allow a t-fraction of possible pairs (a, b) and (p, q) in our quadruples (a, p, b, q). So our incidence problem has tn^2 points and hyperbolas (instead of n^2), and

$$E_t(\mathcal{P}_1, \mathcal{P}_2) = I(\mathcal{P}_t, \mathcal{H}_t) \lesssim (tn^2)^{2/3}(tn^2)^{2/3} = t^{4/3}n^{8/3}.$$



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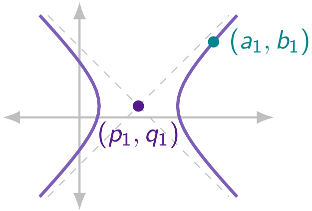


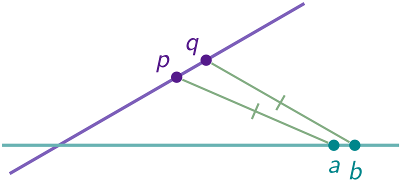
- We might expect that $E_t(\mathcal{P}_1, \mathcal{P}_2) \approx t^2 E(\mathcal{P}_1, \mathcal{P}_2)$. Then we'd get

$$t^2 E(\mathcal{P}_1, \mathcal{P}_2) \lesssim t^{4/3}n^{8/3} \implies E(\mathcal{P}_1, \mathcal{P}_2) \lesssim t^{-2/3}n^{8/3},$$

which would mean proximity doesn't help.

SZ22 — intuition behind proximity

- If p and q are close to each other and $|ap| = |bq|$, then we'd expect a and b to *also* be close to each other.



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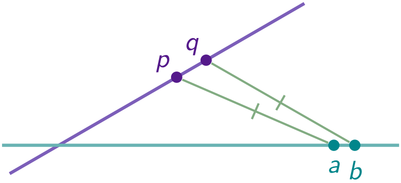

- We can show $E_t(\mathcal{P}_1, \mathcal{P}_2) \gtrsim tE(\mathcal{P}_1, \mathcal{P}_2)$ — shrinking the possibilities for *each* of b and q by a t-fraction only shrinks the number of quadruples with $|ap| = |bq|$ by *one* factor of t, not two.

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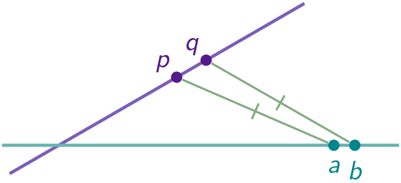

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- Since $E_t(\mathcal{P}_1, \mathcal{P}_2) \lesssim t^{4/3} n^{8/3}$ from the incidence bounds, we get

$$tE(\mathcal{P}_1, \mathcal{P}_2) \lesssim t^{4/3} n^{8/3} \implies E(\mathcal{P}_1, \mathcal{P}_2) \lesssim t^{1/3} n^{8/3},$$

so making t small gives a better bound.

Our result

Theorem

For well-spaced $\mathcal{P}_1$ on a line and $\mathcal{P}_2$ on a strip,

$$\#(\text{distinct distances}) \gtrsim n^{22/15-o(1)}.$$

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For well-spaced $\mathcal{P}_1$ on a line and $\mathcal{P}_2$ on a strip,

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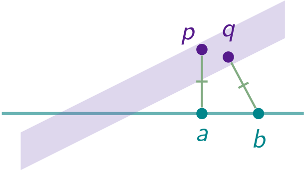
- We also consider the proximal distance energy $E_t(\mathcal{P}_1, \mathcal{P}_2)$.



- We upper-bound $E_t(\mathcal{P}_1, \mathcal{P}_2)$ using another incidence bound.
- We again show $E_t(\mathcal{P}_1, \mathcal{P}_2) \gtrsim tE(\mathcal{P}_1, \mathcal{P}_2)$ — the intuition is the same, though there are a few more details involved.

Acknowledgements

Thanks to Adam Sheffer and Pablo Soberón for organizing this REU, and to Adam Sheffer for his mentorship.

Thanks for listening!