A faster combinatorial algorithm for triangle detection

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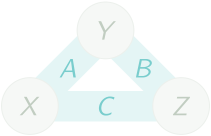
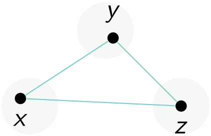
Problem and main result

Problem (Triangle detection)

Given a tripartite graph G, determine whether it has a triangle.

- **Input:** Matrices $A \in \{0,1\}^{X\times Y}$, $B \in \{0,1\}^{Y\times Z}$, $C \in \{0,1\}^{X\times Z}$.
- **Output:** Is there $(x,z) \in X \times Z$ where both AB and C are nonzero?





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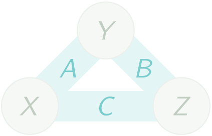

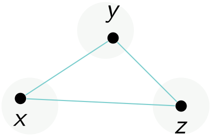

Until recently, the best combinatorial algorithm was $n^3 \cdot (\log n)^{-4}$ time.

Theorem (Abboud–Fischer–Kelley–Lovett–Meka 2024)

There is an $n^3 \cdot 2^{-\Omega((\log n)^{1/7})}$ time combinatorial algorithm for triangle detection (and therefore also BMM).

Notions of regularity

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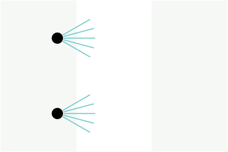
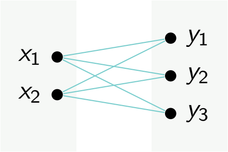
Definition

For $A \in \{0,1\}^{X \times Y}$, we write $\mathbb{E}[A]$ for the density of A. We say A is:

- ε-min-degree if $\deg_A(x) \geq (1-\varepsilon)\mathbb{E}[A]\,|Y|$ for all $x \in X$.
- $(\varepsilon, 2, d)$-grid regular if for random $x_1, x_2 \in X$ and $y_1, \ldots, y_d \in Y$,

$$(\mathbb{P}[x_1, x_2, y_1, \ldots, y_d \text{ form a } K_{2,d}])^{1/2d} \leq (1+\varepsilon)\mathbb{E}[A].$$





Regularity speeds up triangle detection

Theorem (Kelley–Lovett–Meka 2024)

If A and $B^\intercal$ are ε-min-degree and $(\varepsilon, 2, d)$-grid regular, then

$$(1 - 80\varepsilon)\mathbb{E}[A]\mathbb{E}[B]\,|Y| \leq (AB)(x,z) \leq (1 + 80\varepsilon)\mathbb{E}[A]\mathbb{E}[B]\,|Y|$$

for all but a $2^{-\varepsilon d/2}$-fraction of $(x, z) \in X \times Z$.

Think of $\varepsilon \approx \frac{1}{160}$ as a small constant and d as growing with n.

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Think of $\varepsilon \approx \frac{1}{160}$ as a small constant and d as growing with n.

- If A and B meet these conditions, all but a $2^{-\varepsilon d/2}$-fraction of the entries of AB are positive.
- So if more than a $2^{-\varepsilon d/2}$-fraction of the entries of C are 1's, then there is automatically a triangle!
- Otherwise we can brute force: Go through all $(x, z) \in X \times Z$ which are edges in C (there are at most $2^{-\varepsilon d/2}\,|X|\,|Z|$ of these) and all $y \in Y$. This takes $2^{-\varepsilon d/2}\,|X|\,|Y|\,|Z|$ time.

A regularity decomposition

Theorem (AFKLM24)

Given any $A \in \{0,1\}^{X \times Y}$ and $B \in \{0,1\}^{Y \times Z}$, we can decompose

$$AB = \sum_k A_k B_k$$

for smaller matrices $A_k \in \{0,1\}^{X_k \times Y_k}$ and $B_k \in \{0,1\}^{Y_k \times Z_k}$ such that:

- $\mathbb{E}[A_k] \leq 2^{-d}$, $\mathbb{E}[B_k] \leq 2^{-d}$, or both A_k and $B_k^{\intercal}$ are ε-min-degree and $(\varepsilon, 2, d)$-grid regular (for each k).
- $\sum_k |X_k||Y_k||Z_k| \leq \text{poly}(d) \cdot |X||Y||Z|$.
- The number of indices k is at most $2^{O_\varepsilon(d^7)}$.

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Set $d = c(\log n)^{1/7}$ (so $2^{O_\varepsilon(d^7)} \leq n^{0.1}$); this gives runtime

$$\sum_k 2^{-\varepsilon d/2}\,|X_k|\,|Y_k|\,|Z_k| = 2^{-\varepsilon d/2} \cdot \mathrm{poly}(d) \cdot |X|\,|Y|\,|Z| = 2^{-\Omega(d)} n^3.$$

Finding a good rectangle

Lemma

We can decompose $A = \sum_\ell A_\ell$ for $A_\ell \in \{0,1\}^{X_\ell \times Y_\ell}$ such that:

- $\mathbb{E}[A_\ell] \le 2^{-d}$, or A_ℓ is ε-min-degree and $(\varepsilon, 2, d)$-grid regular.
- $\sum_\ell |X_\ell|\,|Y_\ell| \le (d+2) \cdot |X|\,|Y|$.
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Goal

Given $A \in \{0,1\}^{X \times Y}$ such that $\mathbb{E}[A] \geq 2^{-d}$, find a *single* regular piece $A[X_*, Y_*]$ which is not too small (which we call a good rectangle).

Once we can do this, we'll get the decomposition by iteratively removing good rectangles.

Density increments

Main idea: If A is not regular, we can find a density increment — a piece $A[X', Y']$ which is substantially denser.

Claim

Let $\gamma \in (0,1)$. We can find X' with $|X'| \geq (1-\gamma)|X|$ such that either $A[X', Y]$ is ε-min-degree, or $\mathbb{E}[A[X', Y]] \geq (1+\varepsilon\gamma)\mathbb{E}[A]$.

Claim

Either A itself is $(\varepsilon, 2, d)$-grid regular, or we can find X' and Y' with

$$|X'|\,|Y'| \geq \frac{\varepsilon}{16} \cdot \mathbb{E}[A]^{-2d} \cdot |X|\,|Y|$$

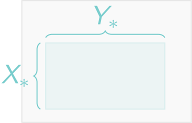
such that $\mathbb{E}[A[X', Y']] \geq (1+\frac{\varepsilon}{2})\mathbb{E}[A]$.

We'll have $\mathbb{E}[A] \geq 2^{-d}$, so this shrinks our sets by $\frac{\varepsilon}{16} \cdot 2^{-2d^2} = 2^{-O_\varepsilon(d^2)}$.

Finding a good rectangle

To find a good rectangle, start with $(X_*, Y_*) \leftarrow (X, Y)$, and repeatedly:

- By shrinking X_* by at most $\frac{1}{2}$, either we can make $A[X_*, Y_*]$ ε-min-degree, or we get a density increment of $(1 + \frac{\varepsilon}{2})$.
- Either $A[X_*, Y_*]$ itself is $(\varepsilon, 2, d)$-grid regular, or we get a density increment of $(1 + \frac{\varepsilon}{2})$ with shrinkage $2^{-O_\varepsilon(d^2)}$.
- If either fails, repeat with this denser submatrix as our new (X_*, Y_*).



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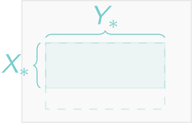

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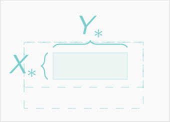

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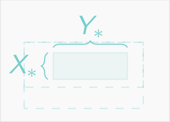

Claim

This gives a good rectangle with $|X_*||Y_*| \geq 2^{-O_\varepsilon(d^3)}|X||Y|$.

- We get a density increment of $(1 + \frac{\varepsilon}{2})$ each time, starting at 2^{-d}, so we fail at most $O_\varepsilon(d)$ times.
- Each failure shrinks $|X_*||Y_*|$ by $2^{-O_\varepsilon(d^2)}$.

Decomposing a single matrix

Lemma

We can decompose $A = \sum_\ell A_\ell$ for $A_\ell \in \{0,1\}^{X_\ell \times Y_\ell}$ such that:

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- Repeatedly find a good rectangle $A[X_*, Y_*]$, add $A_\ell = A[X_*, Y_*]$ to the decomposition, and remove it from A (i.e., update $A \leftarrow A - A_\ell$).
- When $\mathbb{E}[A]$ drops below 2^{-d}, add a final piece to the decomposition consisting of A itself.

1	0	1	0	1
1	1	1	1	0
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Decomposing a single matrix — analysis

To show $\sum_\ell |X_\ell|\,|Y_\ell| \leq (d+2)\cdot|X|\,|Y|$:

- Every piece we remove is at least as dense as the current A, so removing it drops the density of A by a factor of

$$1 - \frac{|X_\ell|\,|Y_\ell|}{|X|\,|Y|}.$$

- The density stays above 2^{-d}, so $\prod_\ell (1 - \frac{|X_\ell||Y_\ell|}{|X||Y|}) \geq 2^{-d}$ (excluding the last two pieces).
- Use the bound $1 - x \leq 2^{-x}$ to conclude.

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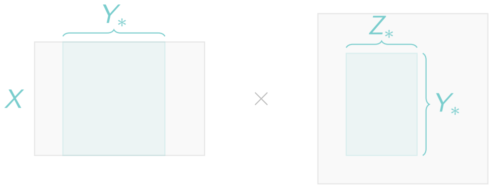
To bound the number of parts:

- Each part has $|X_\ell|\,|Y_\ell| \ge 2^{-O_\varepsilon(d^3)}\,|X|\,|Y|$.
- So there's at most $2^{O_\varepsilon(d^3)}$ indices ℓ in the above sum.

Decomposing a product

Goal

Remove some $B[Y_*, Z_*]$ from B, and account for $A[X, Y_*]B[Y_*, Z_*]$.



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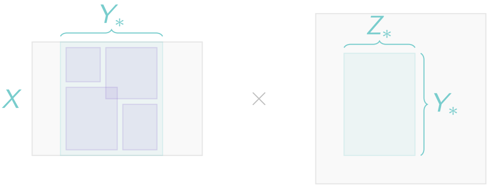

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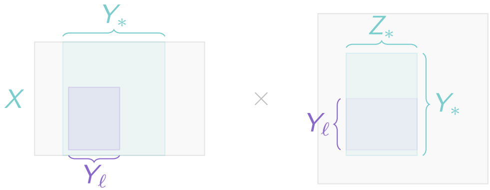

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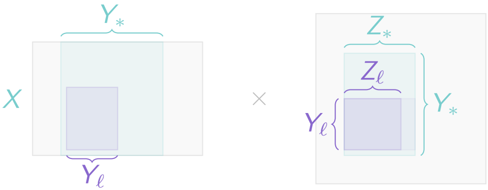

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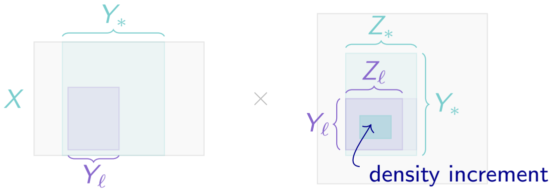

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  - $B[Y_\ell, Z_\ell]^\intercal$ is $(\varepsilon, 2, d)$-grid regular, or we get an increment of $(1+\frac{\varepsilon}{2})$.
- We're missing $A[X_\ell, Y_\ell]B[Y_\ell, Z_* \setminus Z_\ell]$; we recurse to handle it.